ITMO UNIVERSITY

Learning Book

International summer school of Control Systems and Robotics

Saint Petersburg
2019

МИНИСТЕРСТВО НАУКИ И ВЫСШЕГО ОБРАЗОВАНИЯ
РОССИЙСКОЙ ФЕДЕРАЦИИ

УНИВЕРСИТЕТ ИТМО

**О.И. Борисов, А.А. Ведяков, С.М. Власов,
Н.А. Дударенко, К.А. Зименко, А.А. Капитонов,
С.А. Колюбин, А.А. Маргун, М.М. Синетова**

INTERNATIONAL SUMMER SCHOOL OF CONTROL SYSTEMS AND ROBOTICS. LEARNING BOOK. PART 2 / МЕЖДУНАРОДНАЯ ЛЕТНЯЯ ШКОЛА ПО СИСТЕМАМ УПРАВЛЕНИЯ РОБОТОТЕХНИКЕ. УЧЕБНОЕ ПОСОБИЕ. ЧАСТЬ 2

УЧЕБНОЕ ПОСОБИЕ

РЕКОМЕНДОВАНО К ИСПОЛЬЗОВАНИЮ В УНИВЕРСИТЕТЕ ИТМО
по направлению подготовки 15.03.06, 27.03.04
в качестве учебного пособия для реализации основных профессиональных
образовательных программ высшего образования бакалавриата

УНИВЕРСИТЕТ ИТМО

**Санкт-Петербург
2019**

Борисов О.И., Ведяков А.А., Власов С.М., Дударенко Н.А., Зименко К.А., Капитонов А.А., Колюбин С.А., Маргун А.А., Синетова М.М., International summer school of Control Systems and Robotics. Learning book. Part 2 / Международная летняя школа по системам управления робототехнике. Учебное пособие. Часть 2– СПб: Университет ИТМО, 2019. – 216 с.

Рецензент:

Николаев Николай Анатольевич, кандидат технических наук, доцент (квалификационная категория "ординарный доцент") факультета систем управления и робототехники, Университета ИТМО.

The textbook contains theoretical material for studying Control Systems and Robotics. The order of topics follows the structure of the lectures given at ITMO University, Faculty of Control Systems and Robotics. The modern control approaches and digital control systems in robotics are considered. The textbook is intended to foreign students majoring in specializations 27.03.04 Control in Technical Systems and 15.03.06 Mechatronics and Robotics.

Учебное пособие содержит теоретический материал для изучения систем управления и робототехники. Темы в пособии отражают структуру лекций, читаемых в Университете ИТМО на факультете Систем управления и робототехники. В учебном пособии рассматриваются современные подходы к управлению, а также цифровые системы управления в робототехнике. Учебное пособие предназначено для иностранных студентов, обучающихся по направлениям подготовки 27.03.04 Управление в технических системах и 15.03.06 Мехатроника и робототехника.

УНИВЕРСИТЕТ ИТМО

Университет ИТМО – ведущий вуз России в области информационных и фотонных технологий, один из немногих российских вузов, получивших в 2009 году статус национального исследовательского университета. С 2013 года Университет ИТМО – участник программы повышения конкурентоспособности российских университетов среди ведущих мировых научно-образовательных центров, известной как проект «5 в 100». Цель Университета ИТМО – становление исследовательского университета мирового уровня, предпринимательского по типу, ориентированного на интернационализацию всех направлений деятельности.

© Университет ИТМО, 2019

Table of Contents

The modern theory of control systems

Mathematical basics of the systems theory

Mathematical Basics of the Systems Theory

Natalia Dudarenko,
dudarenko@mail.ifmo.ru / dudarenko@yandex.ru

Elementary concepts of matrices

The effectiveness of using matrices in practical calculations is readily realized by considering the solution of a set of linear simultaneous equations, such as

$$\begin{aligned} 5x_1 - 4x_2 + x_3 \qquad &= 0 \\ -4x_1 + 6x_2 - 4x_3 + x_4 &= 1 \\ x_1 - 4x_2 + 6x_3 - 4x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned}$$

Using matrix notations, this set of equations is written as

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{Ax = B}$$

$$A = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2

Special matrices

Definition: A ***matrix*** is an array of ordered numbers. A general matrix consists of mn numbers arranged in m rows and n columns, giving the following array:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \dim A = m \times n$$

- $m = n$ – square matrix;
- $a_{ij} = a_{ji}$ – symmetric matrix;
- $\Lambda = diag\{a_{ij}\}$ – diagonal matrix, a_{ij}=0 for $i \neq j$;
- $I = diag\{1\}$ - identity (or unit) matrix;
- 0 – null matrix;
- A^T - transpose matrix;
- A^{-1} - inverse matrix;
- G^{-H} - Hermitian matrix.

3

Special matrices

Definition: the ***identity*** (or ***unit***) **matrix *I*** is a square matrix of order n with only zero elements except for its diagonal entries, which are unity. For example, the identity matrix of order 3 is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In analogy with the identity matrix, we also use ***identity*** (or ***unit***) **vectors** of order n, defined as $\boldsymbol{e_i}$, where the subscript i indicates that the vector is the ith column of an identity matrix.

Definition: ***null matrix 0*** is a matrix with only zero elements.

Definition: an upper/lower ***triangular matrix*** is a matrix, where below/above the main diagonal are zeros elements

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

4

Special matrices

Definition: the ***diagonal matrix*** Λ is a square matrix of order n with nonzero elements only on the diagonal of the matrix.

$$\Lambda = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

For example, the identity matrix is also a diagonal matrix.

- only one row ($m = 1$) or one column ($n = 1$) – a ***vector***.
  The following are matrices:

$$\begin{bmatrix} 1 & -7.1 \end{bmatrix} \quad \begin{bmatrix} -4 & 0 \\ 9 & 7.6 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -3 \\ 5.1 \end{bmatrix} \quad \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

row vector — **square matrix** — column vector — **symmetric matrix**

5

Special matrices

Definition: The ***transpose*** of the $m \times n$ matrix A, written as $\boldsymbol{A^T}$, is obtained by interchanging the rows and columns in A. If $A = \boldsymbol{A^T}$, it follows that the number of rows and columns in A are equal and that $a_{ij} = a_{ji}$. Because $m = n$, we say that A is a ***square matrix*** of order n, and because $a_{ij} = a_{ji}$, we say that A is a ***symmetric matrix***. *Note that symmetry implies that A is square but not vice versa.*

$$A = \begin{bmatrix} 2 & 0 & 6.7 \\ 5.2 & 3 & -4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 5.2 \\ 0 & 3 \\ 6.7 & -4 \end{bmatrix}.$$

Definition: the ***transpose of the product of two matrices*** A and B is equal to the product of the transposed matrices in reverse order

$$(AB)^T = B^T A^T.$$

6

Matrix multiplication

Definition: *Two matrices A and B can be multiplied* to obtain $C = AB$ if and only if the number of columns in A is equal to the number of rows in B. Assume that A is of order $p \times m$ and B is of order $m \times q$. Then for each element in C we have

$$c_{ij} = \sum_{r=1}^{m} a_{ir}\, b_{rj},$$

where C is of order $p \times q$.

$$\underset{(p \times q)}{C} = \underset{(p \times m)}{A} \times \underset{(m \times q)}{B}$$

Definition: *A matrix is multiplied by a scalar* by multiplying each matrix element by the scalar; i.e., $C = sA$ means that $c_{ij} = sa_{ij}$.

Example. If we consider $D = sI - A$, where s is a scalar, I is a 2×2 identity matrix and A is a 2×2 matrix, then

$$D = sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{bmatrix}.$$

7

Matrix multiplication

- The ***commutative law*** states that matrix multiplication is not commutative:

$$AB \neq BA$$

Although $AB \neq BA$ in general, it may happen that $AB = BA$ for special A and B.

- The ***associative law*** states that

$$(AB)C = A(BC) = ABC$$

In practice, a string of matrix multiplications can be carried out in an arbitrary sequence, and by a clever choice of the sequence, many operations can frequently be saved.

- The ***distributive law*** states that

$$E = (A + B)C = AC + BC.$$

Note that considering the number of operations, the evaluation of E by adding A and B first is much more economical, which is important to remember in the design of an analysis program.

8

Matrix multiplication

Example 1 (*the commutative law*). If we consider the matrices $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $B = [3 \quad 4]$, we have $AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$, $BA = [11]$.

Example 2 (*the associative law*) Calculate A^4, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

1) $A^2 = AA \to A^3 = A^2A \to A^4 = A^3A$.
2) $A^4 = A^2A^2$ and save one matrix multiplication.

Example 3 (*the distributive law*) Evaluate the product v^TAv, where $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$; $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

1) $x = Av \to v^Tx$ (the formal procedure).
2) However, it is more effective to write $A = U + D + U^T$ where U is a lower triangular matrix and D is a diagonal matrix

$U = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$; $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ $\to v^TAv = 2v^TUv + v^TDv$

9

Hadamard product (or Schur product)

Definition: For two matrices A and B of the same dimensions $m \times n$, the **Hadamard product (or Schur product)** $A \circ B$ is a $(m \times n)$ matrix with elements given by

$$(A \circ B)_{ij} = a_{ij} b_{ij}.$$

Then, Hadamard product (or Schur product) $A \circ B$ performs element-by-element multiplication.
The Hadamard product is **commutative**, **associative** and **distributive** over addition:

1) $A \circ B = B \circ A$ 2) $(A \circ B) \circ C = A \circ (B \circ C)$ 3) $(A + B) \circ C = A \circ C + B \circ C$.

Example. Calculate the Hadamard product for two (3×3) matrices A and B.

$$A \circ B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}.$$

10

The trace and determinant of a matrix

Definition: The **trace** of the matrix A is identified as ***tr(A)*** and is equal to $\sum_{i=1}^{n} a_{ii}$, , where n is of the order of A.

$$A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 3 & 6 & 2 & 1 \\ 1 & 2 & 8 & 6 \\ 2 & 1 & 6 & 12 \end{bmatrix} \qquad tr(A) = 4 + 6 + 8 + 12 = 30$$

Definition: The **determinant** of a square matrix A is denoted as **det A** and is defined by the recurrence relation

$$detA = \sum_{i=1}^{n} (-1)^{1+j} a_{1j} detA_{1j}$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by eliminating the $1st$ row and jth column from the matrix A.

A matrix whose determinant is zero is called a ***singular matrix***.

11

The determinant of matrices (the general formulas)

Example 1. Calculate the determinant of A, where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \dim A = 2 \times 2,$$

$$detA = (-1)^2 a_{11} \det A_{11} + (-1)^3 a_{12} \det A_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Example 2. Calculate the determinant of A, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \dim A = 3 \times 3,$$

$$detA = (-1)^2 a_{11} \det A_{11} + (-1)^3 a_{12} \det A_{12} + (-1)^4 a_{13} \det A_{13} =$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

12

The determinant of a matrix

The expansion can be performed by expanding along a row or column using cofactors. The cofactor of an element a_{ij} is the determinant formed by omitting the ith row and jth column.

Example. Calculate the determinant of A, where

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$$

Expanding along the third column, we obtain

$$detA = (-3)\begin{vmatrix} 4 & 5 \\ 7 & -8 \end{vmatrix} - (-6)\begin{vmatrix} 1 & 2 \\ 7 & -8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} =$$

=-3(-32-35)+6(-8-14)+9(5-8)=42

Thus, the cofactor of -3 in the preceding example is the determinant formed by eliminating the first row and third column. The sign is determined from $(-1)^{i+j}$, where i and j are the row and column, respectively, of a_{ij}.

13

Adjoint matrix

Definition: the ***adjoint matrix*** of a square matrix A, denoted by $adjA$ or A^*, is the transpose of its cofactors matrix :

$$\text{adj}(A)_{ij} = \underbrace{(-1)^{i+j} M_{ij}}_{\text{cofactors matrix}}{}^T = (-1)^{i+j} M_{ji}$$

where M_{ij} is the (i,j) *minor* of A, and it is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting row i and column j of A.

Example. Calculate $adjA$, where $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$.

$M_{11} = \det\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = 1$; $A_{11} = (-1)^{1+1} M_{11}{}^T = 1$;

$A_{12} = 0$; $A_{21} = -4$; $A_{22} = 2$;

$$adjA = A^* = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$$

14

The inverse matrix

Definition: The **inverse of a matrix** A is denoted by A^{-1}. Assume that the inverse exists; then the elements of A^{-1} are such that $A^{-1}A = I$ and $AA^{-1} = I$. A matrix that possesses an inverse is said to be *nonsingular*. A matrix without an inverse is a *singular matrix*.

$$A^{-1} = \frac{1}{detA} A^* = \frac{1}{detA} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}$$

Definition: ***the inversion of the product of two matrices*** A and B is equal to the product of the inversed matrices in reverse order

$$(AB)^{-1} = B^{-1}A^{-1}.$$

15

The inverse matrix

Example. Calculate A^{-1}, where $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$.

Solution.

$detA = 2 \cdot 1 - 0 \cdot 4 = 2;$

$A_{11} = 1; A_{12} = 0;$

$A_{21} = -4; A_{22} = 2;$

$$A^* = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$$

Then

$$A^{-1} = \frac{1}{detA} A^* = \frac{1}{2} \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.5 & -2 \\ 0 & 1 \end{bmatrix}$$

Verification:

$$A^{-1} \cdot A = \begin{bmatrix} 0.5 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

16

Hermitian matrix

Definition: the ***Hermitian matrix*** (or self-adjoint matrix), denoted by $\boldsymbol{A^H}$, is a complex square matrix that is equal to its own conjugate transpose. Then, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column for all indices i and j.

Example. Consider a complex square matrix A, where

$$A = \begin{bmatrix} 2 & 1+j & 2-j \\ 1-j & 1 & j \\ 2+j & -j & 1 \end{bmatrix}, (j^2 = -1)$$

Complex conjugate of matrix A is

$$\bar{A} = \begin{bmatrix} 2 & 1-j & 2+j \\ 1+j & 1 & -j \\ 2-j & j & 1 \end{bmatrix}$$

Then, $A^T = \bar{A} \rightarrow A$ is Hermitian matrix. Obviously, the ij-element is conjugate to the ji-element, and we have $A^H = A = \bar{A}^T$.

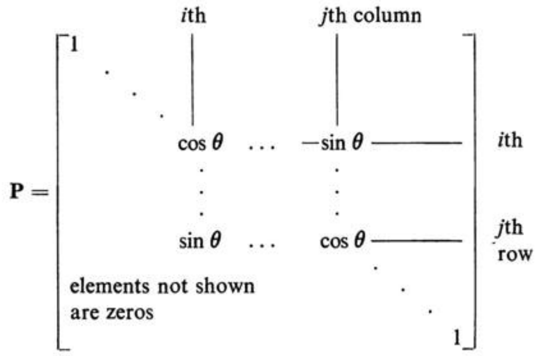
17

Orthogonal matrix

Definition: A matrix P is an ***orthogonal matrix*** if $P^T P = P P^T = I$. Hence, for an orthogonal matrix, we have $P^{-1} = P^T$.

This definition shows that if an orthogonal matrix is used in the change of basis, we have $\bar{A} = P^T A P = P^{-1} A P$. For practical use, there exist some orthogonal matrices that can be constructed and employed easily.

An orthogonal matrix very frequently used is the *rotation matrix*:



$$P = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

18

Orthogonal matrix: rotation matrix

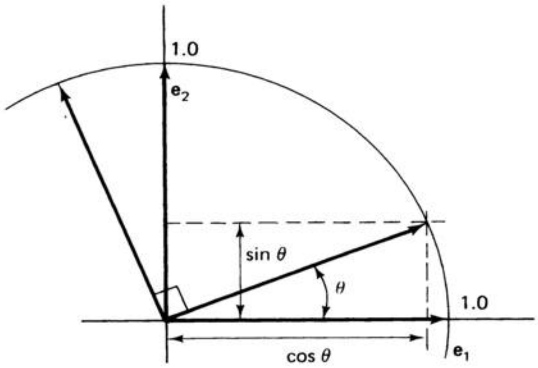


Figure 1.

Referring to figure 1, we observe that for the base vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the transformation $[v_1 \quad v_2] = P[e_1 \quad e_2]$ represents a rotation by an angle θ. In the general case, this rotation is carried out in the n-dimensional space.

19

Orthogonal matrix: rotation matrix

Example. Rotate the vector v through an angle of 45°, where $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The rotation matrix is in this case with $\cos 45° = \sin 45° = \sqrt{2}/2$.

Hence, the rotation matrix is P:

$$P = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

The rotated vector is given by Pv,

$$Pv = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}.$$

It should be noted that the length of the vector Pv is equal to the length of the vector v.

20

Trajectory control of mobile robots

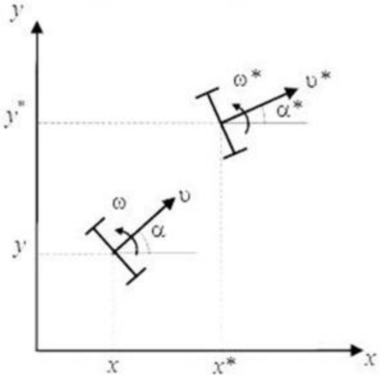


Figure 2.

h - sample rate

$$\begin{cases}\Delta x = hv\cos\alpha,\\ \Delta y = hv\sin\alpha,\\ \Delta\alpha = h\omega,\end{cases}\qquad \begin{cases}\Delta x^* = hv^*\cos\alpha^*,\\ \Delta y^* = hv^*\sin\alpha^*,\\ \Delta\alpha^* = h\omega^*,\end{cases}$$

$$\begin{cases}\Delta(x^*-x) = h(v^*\cos\alpha^* - v\cos\alpha),\\ \Delta(y^*-y) = h(v^*\sin\alpha^* - hv\sin\alpha),\\ \Delta(\alpha^*-\alpha) = h(\omega^*-\omega).\end{cases}$$

$$\begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix} = \begin{pmatrix} T^T(\alpha) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^*-x \\ y^*-y \\ \alpha^*-\alpha \end{pmatrix}$$

$$\boldsymbol{T(\alpha)} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

21

Eigenvalue problem

Any nonzero vector v is eigenvector of a square matrix A, if

$$Av = \lambda v \qquad (1)$$

where λ is ***eigenvalue*** of A.

Each solution consists of an eigen pair, and we write the n solutions as $(\lambda_1, v_1), (\lambda_2, v_2), \dots, (\lambda_n, v_n)$, where $\lambda_1 \le \lambda_2 \le \dots \le \lambda_n$

$$Av_i = \lambda_i v_i$$

The proof that there must be n eigenvalues and corresponding eigenvectors can conveniently be obtained by writing (1) in the form

$$(A - \lambda I)v = 0$$

But these equation only have a solution provided that

$$det(A - \lambda I) = 0 \qquad (2)$$

Using (2), the **eigenvalues** of A are thus the **roots of the polynomial**

$$D(\lambda) = det(A - \lambda I)$$

This polynomial is called the ***characteristic polynomial*** of A.
Since the order of the polynomial is equal to the order of A, we have n eigenvalues, and n corresponding ***eigenvectors***.

22

Eigenvalue problem

Example. Calculate the eigenvalues and eigenvectors of a matrix $A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$.

The characteristic polynomial of A is $p(\lambda) = det(A - \lambda I)$:

$$p(\lambda) = det \begin{bmatrix} -\lambda & 1 \\ -4 & -5-\lambda \end{bmatrix} = \lambda^2 + 5\lambda + 4 = 0$$

The order of the polynomial is 2, and hence there are two eigenvalues:

$$\boldsymbol{\lambda_1 = -1},\ \boldsymbol{\lambda_2 = -4}.$$

To find the eigenvector corresponding to the eigenvalue $\boldsymbol{\lambda_1 = -1}$, we solve:

$$\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

that is,

$$v_2 = -v_1$$
$$-4v_1 - 5v_2 = -v_2.$$

23

Eigenvalue problem

Example (continuation). The two equations are not linearly independent and only one equation is available in two unknowns. Arbitrary choosing $v_1 = 1$, we get $v_2 = -1$. Hence, $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix}$.

For the eigenvalue $\boldsymbol{\lambda_2 = -4}$, we solve:

$$\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

that is,

$$v_2 = -4v_1$$
$$-4v_1 - 5v_2 = -4v_2.$$

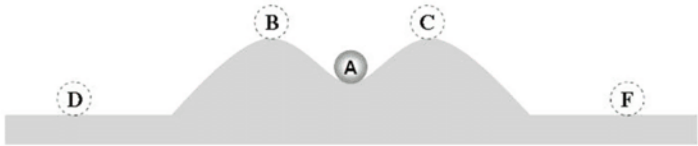
Arbitrary choosing $v_1 = 1$, we get $v_2 = -4$.

Hence, $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-4} \end{bmatrix}$.

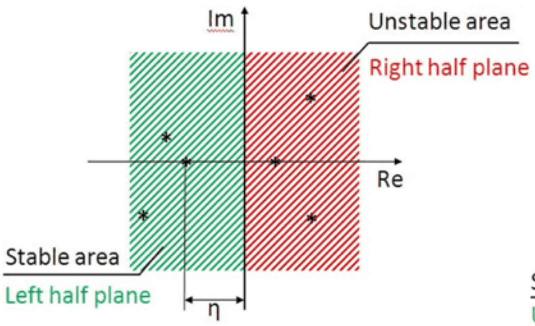
Note:
- $detA = \prod_{i=1}^{n} \lambda_i$;
- $trA = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} A_{ii}$

24

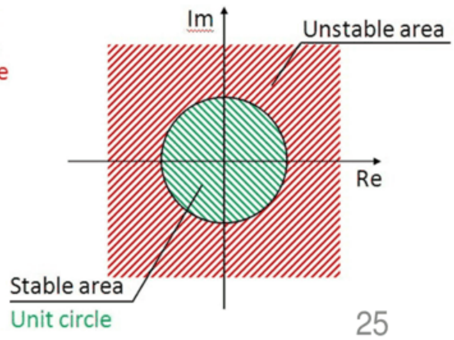
Eigenvalues (poles) and stability of linear systems



Continuous time systems



Discrete time systems



25

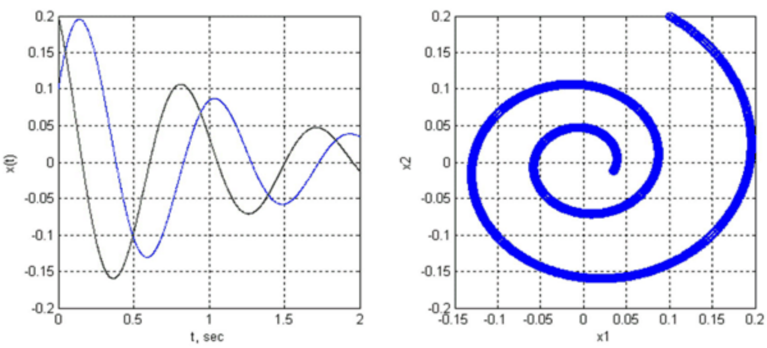
Phase plane. Phase trajectories

The (x_1, x_2) plane is called the **phase plane**. The behavior of the curve relative to the coordinate axes demonstrates the interrelationship between the components $x_1(t)$ and $x_2(t)$ of the solution $x(t)$. Sample solution curves are called **trajectories (phase trajectories)**. Each distinct trajectory is a solution to an initial-value problem.

$$\dot{x}(t) = Ax(t)$$

$$x(t) = e^{At}x(0)$$

$$A = \begin{bmatrix} -2.2 & 7 \\ -7 & -2.2 \end{bmatrix}; \; x(0) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$



26

Phase portrait

A **phase portrait** is a geometric representation of the trajectories of a dynamical system in the phase plane.
The behavior of the phase trajectories (type of equilibrium) is determined by the eigenvalues of the matrix of a system.

Eigenvalues	Type of a phase portrait
Real eigenvalues of the same sign	• NODE
Real eigenvalues of different signs	• SADDLE
Eigenvalues are complex	• SPIRAL SOURCE/SINK
Eigenvalues are imaginary	• CENTER

27

The stability of the origin as determined by the eigenvalues of a (2×2)-matrix A

Собственные числа	Тип устойчивости
$0 < \lambda_1 \leq \lambda_2$	unstable, repelling node
$\lambda_1 < 0 < \lambda_2$	unstable, saddle
$\lambda_1 \leq \lambda_2 < 0$	stable, attracting node
$\lambda_{1,2} = a \pm bi$ and $a > 0$	unstable, spiral source
$\lambda_{1,2} = a \pm bi$ and $a = 0$	stable, center
$\lambda_{1,2} = a \pm bi$ and $a < 0$	stable, spiral sink

28

Phase trajectories

Example. Consider the system of differential equations given by $\dot{x} = Ax$, where $A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$.

1) Compute the eigenvalues and eigenvectors of A and state the general solution to the system.
2) Determine all equilibrium solutions of the system.
3) Plot the phase trajectories.

Solution.

1) Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -4$. Eigenvectors: $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution: $x(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

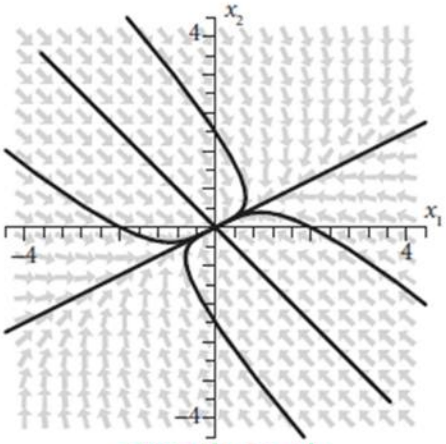
2) The equilibrium solution: $\dot{x} = 0 \rightarrow Ax = 0$. Since A is an invertible matrix, the only solution to $Ax = 0$ is $x = 0$. So the system has the origin as its only equilibrium solution $x = 0$.

3) The origin is a stable equilibrium; 0 is an attracting node.

29

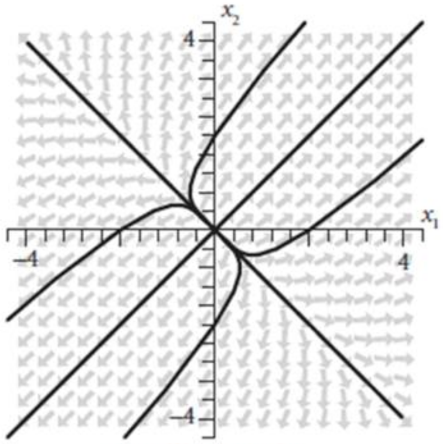
Phase trajectories

$\dot{x} = Ax, A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$.



attracting node

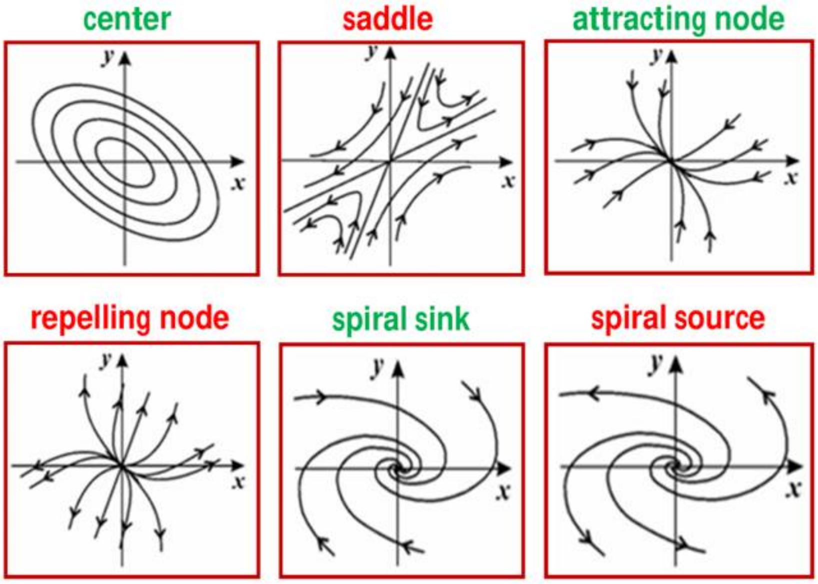
$\dot{x} = Ax, A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.



repelling node

30

Phase portraite



31

Vector norms

Vectors and matrices are functions of many elements, but we shall also need to measure their «**size**». Specifically, if single numbers are used in iterative processes, the convergence of a series of numbers, say $x_1, x_2, \dots x_k$ to a number x is simply measured by

$$\lim_{k\to\infty} |x_k - x| = 0$$

So, convergence is obtained if the residual $y_k = |x_k - x| = 0$ approaches zero as $k \to \infty$. Furthermore, if we can find constants $p \geq 1$ and $c > 0$ such that

$$\lim_{k\to\infty} \frac{|x_{k+1} - x|}{|x_k - x|^p} = c$$

we say that convergence is «of order p». If $p = 1$, convergence is linear and the rate of convergence is c, in which case c must be smaller than 1.

In iterative solution processes using vectors and matrices we also need a measure of convergence.

32

Vector norms

Definition: A ***norm of a vector*** v of order n written as $\|v\|$ is a single number. The norm is a function of the elements of v and the following conditions are satisfied:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$.
2. $\|cv\| = |c|\|v\|$ for any scalar c.
3. $\|v + w\| \leq \|v\| + \|w\|$ for vectors v and w.

A norm is a single number which depends on the magnitude of all elements in the vector or matrix.

The following three vector norms are commonly used, and called the ***infinity***, ***one*** and ***two*** (***Euclidean***) vector norms:

1) $\|v\|_\infty = max_i|v_i|$ 2) $\|v\|_1 = \sum_{i=1}^{n}|v_i|$ 3) $\|v\|_2 = (\sum_{i=1}^{n}|v_i|^2)^{1/2}$

The relationship among the various vector norms:

$\|v\|_\infty \leq \|v\|_1 \leq n\|v\|_\infty$; $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n}\|v\|_\infty$

33

Vector norms

Example. Calculate the 1, 2, and ∞ norms of the vector x, and verify their relationship.

$$x = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

We have

1) $\|v\|_\infty = max_i|v_i| = 3$

2) $\|v\|_1 = \sum_{i=1}^{n}|v_i| = |1| + |-3| + |2| = 1 + 3 + 2 = 6$

3) $\|v\|_2 = (\sum_{i=1}^{n}|v_i|^2)^{1/2} = \sqrt{|1|^2 + |-3|^2 + |2|^2} = \sqrt{14}$

The relationship among the various vector norms:

$\|v\|_\infty \leq \|v\|_1 \leq n\|v\|_\infty$; $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n}\|v\|_\infty$

Hence,

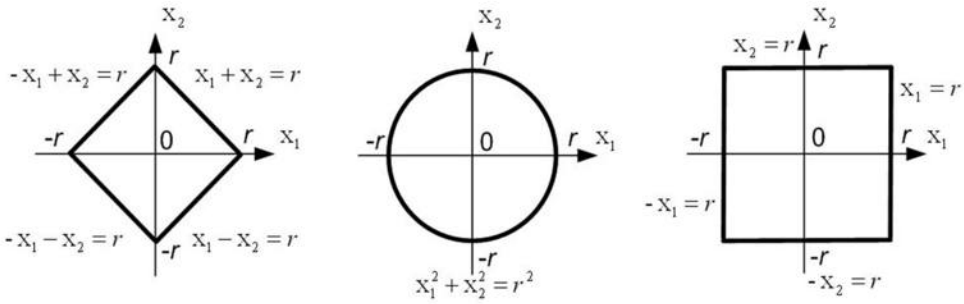
$3 \leq 6 \leq (3)(3) = 9$; $3 \leq \sqrt{14} \leq (\sqrt{3})(3)$;

34

Vector norms

Geometric interpretation of vector norms

$\|x\|_p = r$ for $p = 1, 2, \infty$.



35

Matrix norms

Definition: A ***norm of a matrix*** A of order $n \times n$ written as $\|A\|$ is a single number. The norm is a function of the elements of A and the following relations hold:

1. $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.
2. $\|cA\| = |c|\|A\|$ for any scalar c.
3. $\|A + B\| \leq \|A\| + \|B\|$ for matrices A and B.
4. $\|AB\| \leq \|A\|\|B\|$ for matrices A and B.

The following are frequently used matrix norms:

1) The **infinity or row norm**: $\|A\|_\infty = max_i \sum_{j=1}^{n} |a_{ij}|$;

2) The **column norm**: $\|A\|_1 = max_j \sum_{i=1}^{n} |a_{ij}|$;

3) The **spectral norm**: $\|A\|_2 = \sqrt{\tilde{\lambda}_n}$; $\tilde{\lambda}_n$=maximum eigenvalue of $A^T A$ (for a symmetric matrix A we have $\| \; \|A\|_\infty = \|A\|_1$).

4) The **Euclidian norm**: $\|A\|_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|^2}$

36

Matrix norms

Example. Calculate the 1, 2, and ∞ norms of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{bmatrix}$$

We have

1) $\|A\|_\infty = max\{(|1| + |2| + |3|); (|0| + |-1| + |4|); (|-2| + |5| + |-3|)\} = 10$

2) $\|A\|_1 = max\{(|1| + |0| + |-2|); (|2| + |-1| + |5|); (|3| + |4| + |-3|)\} = 10$

3) $\|A\|_2 = \sqrt{\tilde{\lambda}_n}$

To evaluate $\|A\|_2$ we first need to calculate $A^T A$:

$$A^T A = \begin{vmatrix} 5 & -8 & 9 \\ -8 & 30 & -13 \\ 9 & -13 & 34 \end{vmatrix}.$$

The eigenvalues of $A^T A$ are $\lambda_1 = 1.66$, $\lambda_2 = 18.85$, $\lambda_3 = 48.49$.

Hence we have $\|A\|_2 = \sqrt{\max(\lambda_1, \lambda_2, \lambda_3)} = \sqrt{48.49} = 6.96$

37

Matrix norms

One valuable application of norms arises in the calculation of eigenvalues of a matrix. If we consider the problem $Av = \lambda v$, and take norms on both sides, we obtain

$$\|Av\| = \|\lambda v\|$$

and hence using properties of the norms, we have

$$\|A\|\|v\| \geq |\lambda|\big|\|v\|\big| \text{ or } |\lambda| \leq \|A\|$$

Therefore, every eigenvalue of A is in absolute magnitude smaller than or equal to any norm of A.

Defining the ***spectral radius*** $\rho(A)$ as

$$\rho(A) = max_i|\lambda_i|$$

We have that

$$\rho(A) \leq \|A\|$$

In practice, the ∞ norm of A is calculated most conveniently and thus used effectively to estimate an upper bound on the largest absolute value reached by the eigenvalues.

38

Singular value decomposition (SVD)

Matrix decomposition involves the determination of two or more matrices that, when multiplied in a certain order produce the original matrix.

The ***singular value decomposition*** (***SVD***) of a rectangular matrix A $(m \times n)$ is accomplished by obtaining the matrices U, S, and V, such that

$$A_{m\times n} = U_{m\times m} \cdot S_{m\times n} \cdot V^T{}_{n\times n}$$

where U and V are orthogonal matrices, and S is a diagonal matrix. The diagonal elements of S are called the ***singular values*** of A and are usually ordered so that $\sigma_i \geq \sigma_{i+1}$ for $i=1,2,\ldots,n-1$.

The columns of U and of V are the corresponding ***singular vectors***. *Orthogonal matrices are such that* $U \cdot U^T = I, V \cdot V^T = I$.

MATLAB code for a matrix X : $[U,S,V]$ = SVD(X)

39

Geometric interpretation of svd(A)

$\eta = N\chi$ ⇒ $\|\eta\| = \|N\chi\|$ $\sigma_m\|\chi\| \leq \|\eta\| \leq \sigma_1\|\chi\|$

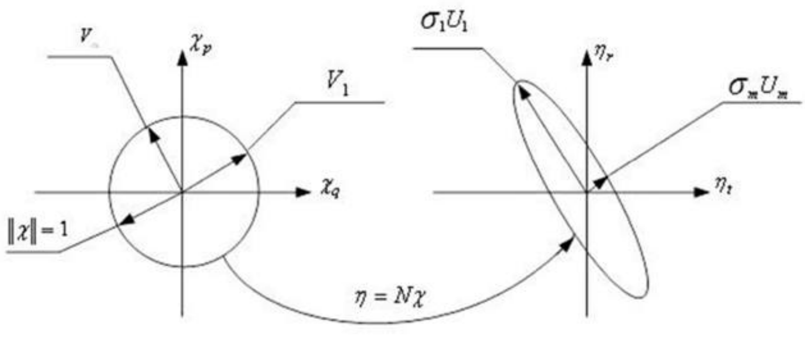


Figure 3.

40

Singular values

Singular values σ_i of any matrix A can be calculated as following

$$\det(\mu I - AA^T) = \det(\mu I - A^T A) = 0; \; \sigma_i = |\mu_i{}^{1/2}|$$

Example. Calculate the singular values of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 5 & -3 \end{bmatrix}$$

We first need to calculate $A^T A$:

$$A^T A = \begin{vmatrix} 5 & -8 & 9 \\ -8 & 30 & -13 \\ 9 & -13 & 34 \end{vmatrix}.$$

The eigenvalues μ_i of $A^T A$ are the following

$$\mu_1 = 48.49, \; \mu_2 = 18.85, \; \mu_3 = 1.66.$$

Hence we have singular values of the matrix A

$$\sigma_1 = \sqrt{48.49}, \; \sigma_2 = \sqrt{18.85}, \; \sigma_3 = \sqrt{1.66}$$

$$\max(\sigma_i) = \sigma_1 = \|A\|_2$$

41

Condition number

The condition number of a square non-singular matrix is defined as the product of the matrix norm times the norm of its inverse

$$\boldsymbol{cond(A) = \|A\| \|A^{-1}\|}$$

It is also defined as the ratio of the largest to the smallest singular values of the matrix

$$\boldsymbol{cond(A) = \frac{\max(\sigma_i)}{\min(\sigma_i)}}$$

The condition number of a non-singular matrix is a measure of how close the matrix is to being singular. The larger the value of the condition number, the closer it is to singularity.

This information will be useful in the analysis of the solution of a linear system.

42

Functions of matrices

- ***Scalar functions of matrices:***
  $f(A) = detA = \prod_{i=1}^{n} \lambda_i$;
  $f(A) = trA = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} A_{ii}$
- ***Vector functions of matrices:***
  $f(A) = \{col(\lambda_i, i = \overline{1,n})\}$ - vector of eigenvalues;
  $f(A) = \{col(a_i, i = \overline{1,n})\}$ - vector of coefficients of the characteristic polynomial $D(\lambda)$;
- ***Matrix functions of matrices:***
  any scalar row $f(\alpha) = a_0 + a_1\alpha^1 + a_2\alpha^2 + ... + a_p\alpha^p + ... \rightarrow$
  matrix function $f(A)$ of a matrix A
  $f(A) = a_0 I + a_1 A^1 + a_2 A^2 + ... + a_p A^p + ...$

43

Functions of matrices

Cayley-Hamilton theorem.
Any square matrix A with dimension n and with a characteristic polynomial

$$D(\lambda) = det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0$$

satisfies its own characteristic equation, that

$$D(A) = A^n + a_{n-1}A^{n-1} + ... + a_0 I = 0.$$

The proof of the theorem is based on the representation of any matrix A with the eigenvalues λ_i in the following form

$$A = M\Lambda M^{-1}$$

where M is the eigenvectors matrix of A, $\Lambda = diag\{\lambda_i, \overline{i = 1, n}\}$ is the diagonal matrix.

$D(A) = M(\Lambda^n + a_{n-1}\Lambda^{n-1} + ... + a_0 I)M^{-1} = 0$

$D(A) = M(diag\{\underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0), \overline{i = 1, n}}_{=0}\})M^{-1} = 0$

44

Functions of matrices

Statement 1. Function of a matrix satisfies its ratio of similarity

$$M\bar{A} = AM \rightarrow Mf(\bar{A}) = f(A)M$$

where $\bar{A}$ is a similarity matrix (for example, the canonical form of A).

Statement 2. Any square matrix A with the spectrum of eigenvalues $\lambda_i, i = \overline{1,n}$ and the eigenvectors $v_i, i = \overline{1,n}$ satisfies its function of the matrix $f(A)$ with the spectrum of eigenvalues λ_{fi}

$$\lambda_{fi} = f(\lambda_i) : \det(\lambda_{fi}I - f(A)) = 0, i = \overline{1,n}$$

and the eigenvectors $v_{fi} = v_i, i = \overline{1,n}$

$$f(A)v_i = f(\lambda_i)v_i.$$

Statement 3. Any power of a matrix A, denoted as A^l, is commutative

$$A^l f(A) = f(A)\, A^l$$

45

Functions of matrices

Example. Calculate function of matrix $f(A) = A^2$, where

$A = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}, \lambda_1 = 1, \lambda_2 = -5, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}.$

- the formal procedure: $f(A) = A^2 = AA = \begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & -3 \end{bmatrix} =$
  $= \begin{bmatrix} 9 & -16 \\ -8 & 17 \end{bmatrix}$
- using the property of similarity transformation

$A = M\Lambda M^{-1} \rightarrow A^2 = M\Lambda^2 M^{-1}$,

where $M = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}, \Lambda^2 = \begin{bmatrix} 1 & 0 \\ 0 & 25 \end{bmatrix} \rightarrow$

$$A^2 = M\Lambda^2 M^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 25 \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 9 & -16 \\ -8 & 17 \end{bmatrix}.$$

46

Functions of matrices

Example. Calculate function of matrix $f(A) = A^2$, where

$f(A) = \begin{bmatrix} 9 & -16 \\ -8 & 17 \end{bmatrix}, \lambda_1 = 1, \lambda_2 = -5, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- the formal procedure:

$$\det\left(\lambda_{fi} I - f(A)\right) = \det \begin{bmatrix} \lambda_{fi} - 9 & 16 \\ 8 & \lambda_{fi} - 17 \end{bmatrix} = {\lambda_{fi}}^2 - 26\lambda_{fi} + 25 = 0$$

$$\boldsymbol{\lambda_{f1} = 1; \lambda_{f2} = 25}.$$

- using the statement 2:

$$\lambda_{fi} = f(\lambda_i) = {\lambda_i}^2: \quad \boldsymbol{\lambda_{f1}} = (1)^2 = \boldsymbol{1}; \boldsymbol{\lambda_{f2}} = (-5)^2 = \boldsymbol{25}.$$

For the eigenvectors:

$$\boldsymbol{f(A)v_1} = \begin{bmatrix} 9 & -16 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boldsymbol{f(\lambda_1)v_1}$$

$$\boldsymbol{f(A)v_2} = \begin{bmatrix} 9 & -16 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 25 \\ -25 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boldsymbol{f(\lambda_2)v_2}$$

47

Examples of matrix functions of matrix

1. $e^{\alpha} = 1 + \alpha + \frac{1}{2!}\alpha^2 + \frac{1}{3!}\alpha^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{t!}\alpha^i$

$$f(A) = e^A = \exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{t!}A^i$$

2. $\ln\alpha = 2\left(\frac{\alpha-1}{\alpha+1}\right) + \frac{2}{3}\left(\frac{\alpha-1}{\alpha+1}\right)^3 + \frac{2}{5}\left(\frac{\alpha-1}{\alpha+1}\right)^5 + \ldots$

$\ln A = 2(A - I)(A + I)^{-1} + \frac{2}{3}(A - I)^3(A + I)^{-3} + \frac{2}{5}(A - I)^5(A + I)^{-5} + \ldots$

3. $\cos\alpha = 1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 - \frac{1}{6!}\alpha^6 \ldots$

$$\cos A = 1 - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \frac{1}{6!}A^6 \ldots$$

4. $\sin\alpha = 1 - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 - \frac{1}{7!}\alpha^7 \ldots$

$$\sin A = 1 - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \frac{1}{7!}A^7 \ldots$$

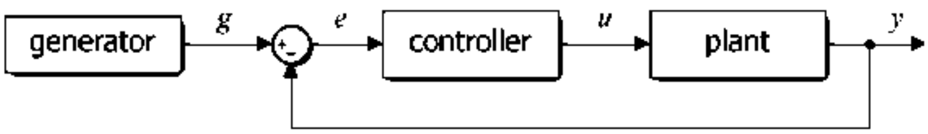
48

Basics of the control theory

Basics of Automatic Control Theory

Madina Sinetova

Structural scheme



- **Plant (Control Object) - physical device (DC motor, electrical circuit, combustion engine, etc.),**
- $e(t) = g(t) - y(t)$ **- error signal,**
- $u(t)$ **- control signal,**
- $g(t)$ **- reference signal,**
- $y(t)$ **- output signal (for example, motor shaft velocity or rotation angle).**

2

Input-State-Output model

Model is:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases}$$

- $x \in R^n$ – state vector; $A -$ $n \times n$ «state matrix»;
- $u \in R^k$ – system input; $B - n \times k$ «input matrix»;
- $y \in R^l$ – system output; $C - l \times n$ «output matrix».

Let's introduce algebraic variable s and introduce *characteristic equation:*

$$\det(A - sI) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0,$$

I – identity matrix,

$s_i, i = \overline{1, n}$ are roots of the system,

$a_j, j = \overline{0, n-1}$ are polynomial coefficients.

3

Input-Output model

System behavior is described by one differential equation of n-order:

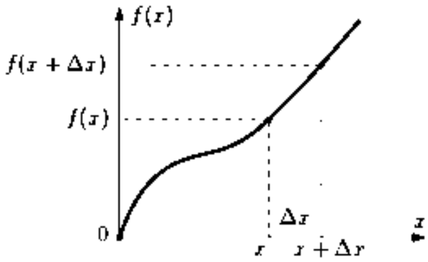
$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y =$$
$$= b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u,$$

- $a_i, i = \overline{0, n-1},$
- $b_j, j = \overline{0, m}.$

Used two coefficients sets instead of matrices A, B, C.

4

Physical feasibility



$m \leq n$ – *physical feasibility condition.*

Derivative is a:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In *physical* systems $x \to t$.

We can't know $f(t + \Delta t)$ value, because the moment $(t + \Delta t)$ in the future.

5

Transfer function

Let's introduce algebraic variable $s = \frac{d}{dt}$ as a *differentiation operator.*

Rewrite differential equation of n-order:

$$s^n y + a_{n-1} s^{n-1} y + \cdots + a_1 s y + a_0 y =$$
$$= b_m s^m u + b_{m-1} s^{m-1} u + \cdots + b_1 s u + b_0 u.$$

Variables y and u put beyond the bracket:

$$y(\underbrace{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}_{\text{characteristic equation}}) =$$
$$= u(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0).$$

Divide equation above to u and to characteristic equation:

$$\frac{y}{u} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = W(s).$$

6

Transfer function

$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$ – *Transfer function.*

- Roots of the Transfer function denominator are called *poles* and described system's *free* motion.
- Roots of the Transfer function numerator are called *zeros* and described system's *forced* motion.

7

Example

Given:

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{du(t)}{dt} + u(t).$$

Find:

$$W(s) = \frac{y(s)}{u(s)} - ?$$

Solution:

- *satisfies the physical feasibility condition:* $m=1 < n = 2$,
- $W(s) = \frac{s+1}{s^2+2s+1}$,
- $s + 1 = 0 \Rightarrow s = -1$ – zero;
- $s^2 + 2s + 1 = 0 \Rightarrow s_{1,2} = -1$ – two poles (second order pole).

8

Conversion ISO to IO

$$W(s) = C(A - sI)^{-1}B$$

- MIMO case:

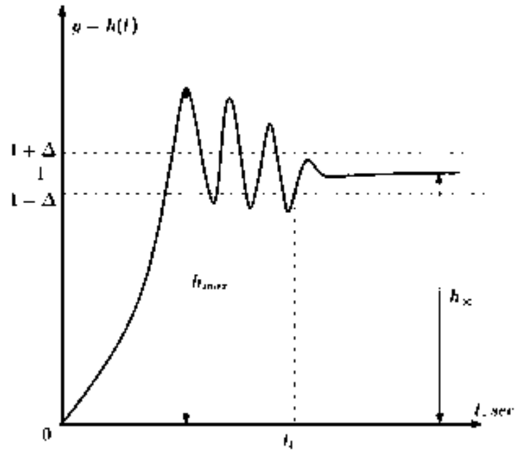
  Transfer matrix $l \times k$ dimension, consists of several transfer functions $W_{i,j}(s)$ linking i-th output and j-th input.

- SISO case:

  Transfer function $W(s)$.

9

Transients



- t_t – transient time: $\forall t > t_t$: $|h(t) - 1| < \Delta, \Delta > 0$;
- $h(t)$ – transient function;
- $\Delta = 0.05$ (5% deviation from steady value h_∞);
- $h_\infty = \lim_{t \to \infty} h(t)$;
- In case $g(t) = \text{const} = g$: $h_\infty = W(s)|_{s=0} \cdot g$;
- $\delta = \left|\frac{h_{max} - h_\infty}{h_\infty}\right| \cdot 100\%$ - overcontrol (typically $\delta = 0 \cdots 30\%$);

10

Example

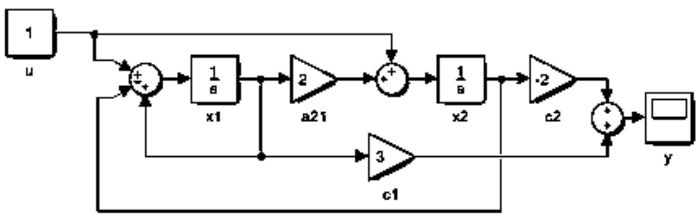
- Given:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; C = [3 \quad -2].$$

- Differential equations system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_1 - x_2 + u \\ \dot{x}_2 = 2x_1 + 2u \\ y = 3x_1 - 2x_2 \end{cases}.$$

- Modelling:



11

Example

- Transfer function:

$$W(s) = C(A - sI)^{-1}B;$$

$$A - sI = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} 1 - s & 1 \\ -2 & 1 - s \end{bmatrix} = A^*;$$

$$A^{*-1} = \frac{1}{\det A^*}(\mathrm{adj} A^*)^{\mathrm{T}};$$

$\mathrm{adj} A^*$ – *adjoint* (or allied or interconnected) matrix;

$a_{ij}^* = (-1)^{i+j} \cdot M_{ij}$ – *algebraic complements*,

i – row number, j – column number, M_{ij} – minors of A^*.

$\det A^* = (1 - s)(-s) - (2)(-1) = -s + s^2 + 2 = s^2 - s + 2.$

12

Example

- $A^{*-1} = \frac{1}{s^2-s+2}\begin{bmatrix} -s & -2 \\ 1 & 1-s \end{bmatrix}^{\mathrm{T}} = \frac{1}{s^2-s+2}\begin{bmatrix} -s & 1 \\ -2 & 1-s \end{bmatrix};$
- $CA^{*-1} = [3 \quad -2] \cdot \frac{1}{s^2-s+2} \cdot \begin{bmatrix} -s & 1 \\ -2 & 1-s \end{bmatrix} = \frac{1}{s^2-s+2} \cdot [-3s+4 \quad 3-2+2s];$
- $W(s) = \frac{1}{s^2-s+2} \cdot [-3s+4 \quad 1+2s] \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$
  $= \frac{1}{s^2-s+2}(-3s+4+2+4s) = \frac{s+6}{s^2-s+2}.$

13

Example

- Transfer from IO to ISO:

$$W(s) = \frac{Y(s)}{U(s)} = \frac{s+6}{s^2-s+2}, \; s = \frac{d}{dt}.$$

$$s^2y - sy + 2y = su + 6u.$$

- Divide the equation to s^n:

$$s^2y - sy + 2y = su + 6u \mid : s^2,$$

$$y - \frac{1}{s}y + 2\frac{1}{s^2}y = \frac{1}{s}u + 6\frac{1}{s^2}u.$$

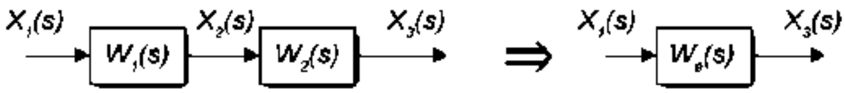
- Leave only y on the left:

$$y = \frac{1}{s}u + \frac{1}{s}y + 6\frac{1}{s^2}u - 2\frac{1}{s^2}y.$$

14

Structural transformations

1. Consecutively k-connected elements:



$$W_e(s) = \frac{X_3(s)}{X_1(s)}.$$

- From the scheme :

$$\begin{cases} X_2 = W_1 X_1, \\ X_3 = W_2 X_2 = W_2 W_1 X_1 \end{cases} \Rightarrow W_e(s) = \frac{X_3(s)}{X_1(s)} = W_2(s) W_1(s).$$

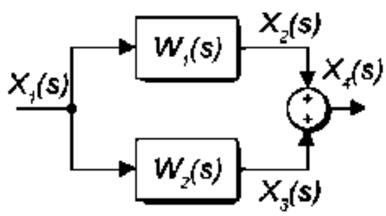
- So:

$$W_e(s) = \prod_{i=1}^{k} W_i(s).$$

15

Structural transformations

2. Parallel k-connected elements:



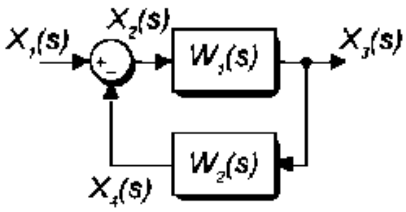
- With the same logic:

$$W_e(s) = \sum_{i=1}^{k} W_i(s).$$

16

Structural transformations

3. Elements with a feedback:



$$\begin{cases} X_2 = X_1 - X_4, \\ X_3 = W_1 X_2, \\ X_4 = W_2 X_3, \end{cases} \Rightarrow \begin{cases} X_3 = W_1(X_1 - X_4), \\ X_3 = W_1 X_1 - W_1 W_2 X_3, \\ (1 + W_1 W_2) X_3 = W_1 X_1. \end{cases}$$

$$W_e(s) = \frac{X_3}{X_1} = \frac{W_1}{1 + W_1 W_2}.$$

17

Stability

- *Stability* is the system ability to return to initial position after stopping action to system external disturbances.

Stable system Unstable system

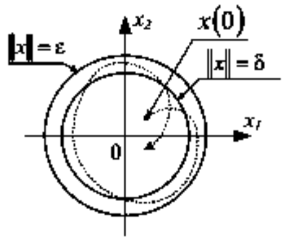
Segway

18

Stability types

1. Lyapunov stability.

- Guarantees bounded of all trajectories, but not guarantees convergence to some steady value.



where x_1, x_2 are state coordinates, ε, δ – some small numbers; as norm of x_1 and x_2 can be used quadratic norm, for example; $x(0)$ – initial position of trajectory.

- The equilibrium $x = 0$ is *Lyapunov stable* if for any small number $\varepsilon > 0$, exists small number $\delta(\varepsilon) > 0$, that for all trajectories starting from the initial conditions $\|x(0)\| \le \delta(\varepsilon)$ for any time $\forall t \ge 0$ following inequality is satisfied: $\|x(t)\| \le \varepsilon$.

19

Root stability criterion

Given:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}.$$

Characteristic polynomial of the given system:

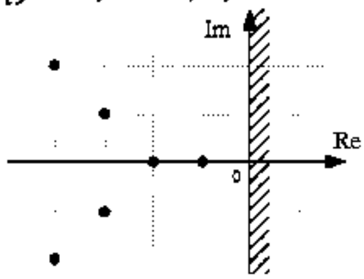
$$\det(A - sI) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0,$$

where $s_i, i = \overline{1, n}$ – roots of the polynomial.

If all roots have *negative real parts* $\mathrm{Re}(s_i) < 0, i = \overline{1, n}$, then the system is stable.

Im – imaginary axis (stability border),

Re – real axis.



20

Stability borders

If one or more than one root is more than zero system is *unstable*.

1.1. Stability border of neutral type.

Dynamic system is on the border of neutral type if one or two roots of characteristic polynomial are equal to zero and rest roots have negative real parts:

$$s_{1,2} = 0, \mathrm{Re}(s_i) < 0, i = \overline{3,n}.$$

21

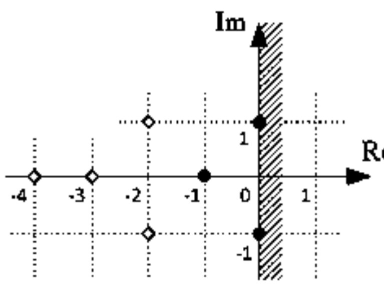
Stability borders

1.2. Stability border of oscillatory type.

The dynamic system is on the border of oscillatory type if the characteristic polynomial has pair of purely imaginary roots and rest roots have negative real parts:

$$s_{1,2} = \pm j\omega, \omega > 0\ \mathrm{Re}(s_i) < 0, i = \overline{3,n},$$

where j – imaginary unit.

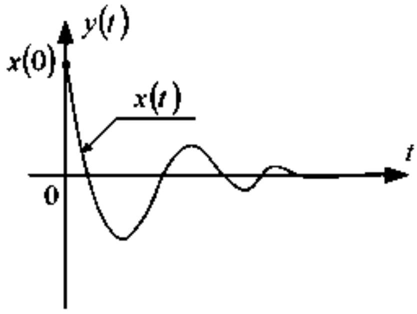


22

Stability types

2. Asymptotic stability

The equilibrium $x = 0$ is *asymptotically stable* if the position is Lyapunov stable and for any motion trajectories $x(t)$ from the arbitrary initial conditions $x(0)$ the condition $\lim_{t\to\infty} \|x(t)\| = 0$.

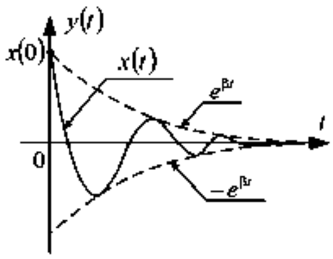


23

Stability types

3. Exponential stability

The equilibrium $x = 0$ is *exponential stable* if for any motion trajectories $x(t)$ from the arbitrary initial conditions $x(0)$ exists numbers $\beta < 0$ and $\rho \geq 1$ that for any time $\forall t \geq 0$ the inequality is satisfied: $\|x(t)\| \leq \rho e^{\beta t} \cdot \|x(0)\|$.



Constant β is the convergence degree and characterizes convergence velocity to equilibrium.

24

Modal control

Given plant:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}.$$

Modal (reference) model is an autonomous dynamic system:

$$\begin{cases} \dot{z}(t) = \Gamma z(t) \\ \eta(t) = \mathrm{H}z(t) \end{cases}'$$

where $z(t) \in R^n$ – state vector, Γ – $n\times n$ state matrix;

$\eta(t) \in R^l$ – output vector; H – $l\times n$ output matrix.

Matrices (Γ, H) – completely observable.

Matrix Γ characterized by eigenvalues $\lambda_i, i = \overline{1,n}$.

25

Modal control

Choose proportional control :

$$u(t) = -\mathrm{K}x(t),$$

where K – matrix of linear stationary feedbacks.

Substituting control signal $u(t)$:

$$\begin{cases} \dot{x} = \mathrm{F}x \\ y = Cx \end{cases}'$$

where $\mathrm{F} = A - B\mathrm{K}$ – matrix of the closed system.

- To matrix K provides quality indicators for the given dynamic system like in a reference model, it's necessary the condition of similarity is satisfied:

$$x(t) = \mathrm{M}z(t) \Rightarrow z(t) = \mathrm{M}^{-1}x(t), t \geq 0,$$

where M – *coordinate transformation* or *similarity matrix.*

26

Modal control

Output of the reference model is a control signal for the given model. Using this relation obtain the control law:

$$u(t) = -\mathrm{H}z(t) = -\mathrm{H}\mathrm{M}^{-1}x(t).$$

- Let's introduce notation:

$$\mathrm{K} = \mathrm{H}\mathrm{M}^{-1} \Leftrightarrow \mathrm{H} = \mathrm{K}\mathrm{M}.$$

- Substituting obtained expressions to Sylvester type matrix equation obtain:

$$\mathrm{M}\Gamma - A\mathrm{M} = -B\mathrm{K}\mathrm{M}.$$

- With the notation $\mathrm{F} = A - B\mathrm{K}$ matrix equation leads to a similarity condition:

$$\mathrm{M}\Gamma = \mathrm{F}\mathrm{M},$$

therefore matrix F has eigenvalues Γ.

27

Example

Given plant:

$$\mathrm{A} = \begin{bmatrix} 7 & 3 & 14 \\ 6 & 5 & -8 \\ 4 & -1 & -7 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathrm{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Required to ensure the transition time in a closed system $t_t = 1.7$ seconds.

```
A = [7 3 14; 6 5 -8; 4 -1 -7]; % MATLAB code
B = [0; 0; 1];
C = [1 0 0];
```

28

Example

1. Checking system's stability:

$$\det(A - sI) = s^3 - 5s^2 - 131s + 635 \Rightarrow \begin{cases} s_1 = -11.3921, \\ s_2 = 11.5776 > 0, \\ s_3 = 4.8145 > 0, \end{cases}$$

$\Rightarrow$ system is unstable.

```
roots(poly(A)); % MATLAB code or
eig(A);
```

29

Example

2. Checking system for complete controllability (matrix $Nc = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$ should has the full rank):

$$\text{rank } \mathrm{N}_c = \text{rank} \begin{bmatrix} 0 & 14 & -24 \\ 0 & -8 & 100 \\ 1 & -7 & 113 \end{bmatrix} = 3 = n \Rightarrow$$ system is completely controllable.

```
rank(ctrb(A, B)); % MATLAB code
```

30

Example

3. Form reference model:

- Find a desired characteristic polynomial in accordance with a normalized transient time t_t^* for a n-order system.
- It can be calculated from transients of Butterworth (overcontrol not more than 15%):

$$Db(s) = \prod_{i=1}^{n}\left(s - \omega e^{j\left(\frac{\pi}{2}+\frac{2i-1}{2n}\pi\right)}\right),$$

or Newton (overcontrol 0%):

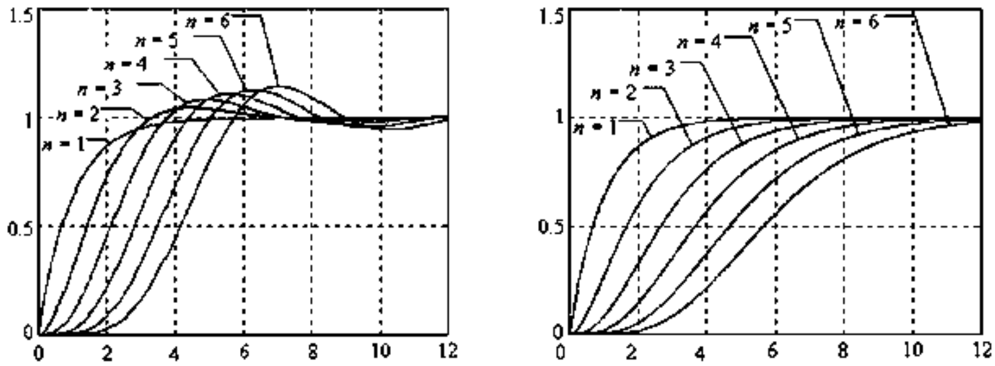
$$Dn(s) = (s+\omega)^n$$

polynomials.

31

Example

3. Form reference model:



Butterworth polynomial transient. Newton polynomial transient.

32

Example

3. Form reference model:

- In case $n = 3$: $t_t^* = 6.2$ seconds:

$$D_d(s) = s^3 + 3\omega s^2 + 3\omega^2 s + \omega^3$$

is the desired characteristic polynomial (third-order Newton polynomial).

$$\omega = \frac{t_t^*}{t_t} = \frac{6.2}{1.7} = 3.65,\ D_d(s) = s^3 + 10.94s^2 + 39.9s + 48.51.$$

- Matrix Γ of the reference model in a *canonical controllable form* with the desired polynomial:

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48.51 & -39.9 & -10.94 \end{bmatrix}, \mathrm{H} = [1 \quad 0 \quad 0].$$

33

Example

```
H = [1 0 0]; % MATLAB code
G = [0 1 0; 0 0 1; -48.51 -39.9 -10.94];
```

4. Finding matrix transformation M:

- The solution of the Sylvester matrix equation with respect to the matrix M:

$$\mathrm{M} = \begin{bmatrix} -0.1497 & -0.0084 & -0.0020 \\ 0.2111 & 0.0215 & 0.0028 \\ 0.0365 & -0.0054 & 0.0014 \end{bmatrix}.$$

```
M = sylv(-A,G,-B*H); % MATLAB code
```

34

Example

5. Calculation of matrix K:

$$K = HM^{-1} = [36.2969 \quad 18.2453 \quad 15.94].$$

```
K = H * inv(M); % MATLAB code
```

6. Checking calculations:

$$F = A - BK = \begin{bmatrix} 7 & 3 & 14 \\ 6 & 5 & -8 \\ -32.2959 & -19.2453 & -22.94 \end{bmatrix},$$

$$\det(F - sI) = s^3 + 10.94s^2 + 39.9s + 48.51 = D_d(s).$$

Characteristic polynomial are the same with the reference, hence controller coefficients found correctly.

```
F = A – B * K; % MATLAB code
poly(F)
```

35

Example

7. Calculation of the direct linking coefficient:

$$K_g = -(C(A - BK)^{-1}B)^{-1} = -0.5161.$$

```
Kg = -inv(C * inv(A – B * K) * B); % MATLAB code
```

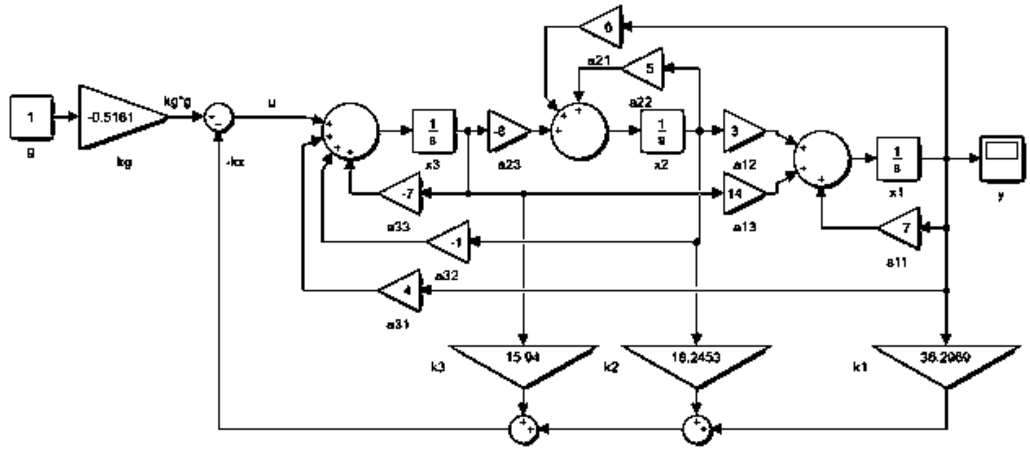
8. Form control signal:

$$u(t) = K_g g(t) - Kx(t).$$

36

Example

9. Modelling:



37

Discretizing

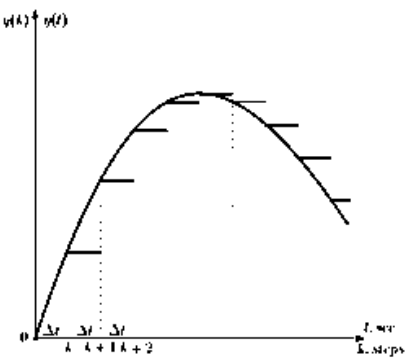
- In continuous time, the Plant is described as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}.$$

- Digital electronic microcontrollers work in a discrete time, so:

$$\begin{cases} x(k+1) = Ad \cdot x(k) + Bd \cdot u(k) \\ y(k) = Cd \cdot x(k) \end{cases},$$

$T = \Delta t$ – discrete interval, k – steps.



38

Discretizing

Discrete matrix Ad can be found as a matrix exponent:

$$Ad = e^{AT}.$$

Discrete matrix Bd we can find using formula:

$$Bd = A^{-1}(e^{AT} - I)B|_{\exists A^{-1}},$$

and

$$Cd = C.$$

```
T = 0.1; % MATLAB code
Ad = expm(A * T);
Bd = inv(A) * (Ad - eye(n)) * B;
Cd = C;
```

39

Discretizing

- To avoid division by zero in A^{-1}, matrix exponent can be decomposed to series with k-members:

$$e^{AT} \approx \sum_{i=0}^{k} \frac{A^i T^i}{i!} = Ad,$$

$$Bd \approx \left(\sum_{i=1}^{k} \frac{A^{i-1} T^i}{i!} \right) \cdot B.$$

40

Example

Given:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, T = 0.01, u = 1.$$

Find:

$$Ad, Bd, Cd.$$

Code:

```
A = [0 1; -2 -3]; % MATLAB code
B = [0; 1];
C = [1 0];
T = 0.01;
Ad = 0;
Bd = 0;
k = 10;
```

41

Example

```
for i = 0:1:k
Ad = Ad + (A^i * T^i / factorial(i));
if(i > 0)
      Bd = Bd + (A^(i – 1) * T^i /
            factorial(i)) * B;
end;
end;
Cd = C;
```

$$Ad = \begin{bmatrix} 0.9999 & 0.0099 \\ -0.0197 & 0.9703 \end{bmatrix}, Bd = \begin{bmatrix} 0 \\ 0.0099 \end{bmatrix}, Cd = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

42

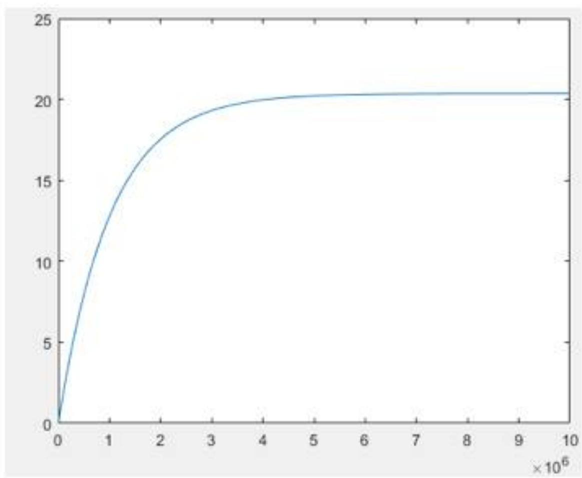
Example

Integrator – memory element which saves previous value, so:

```
x1 = 0; % MATLAB code
x2 = 0; % initial conditions
u = 1;
for m = 0:1:100000
        x1dot = Ad(1,1) * x1 + Ad(1,2) * x2 + Bd(1) * u;
        x2dot = Ad(2,1) * x2 + Ad(2,2) * x2 + Bd(2) * u;
        y(m) = Cd(1) * x1 + Cd(2) * x2;
        k(m) = m / T;
        x1 = x1dot;
        x2 = x2dot;
end;
plot(k,y);
```

43

Example



44

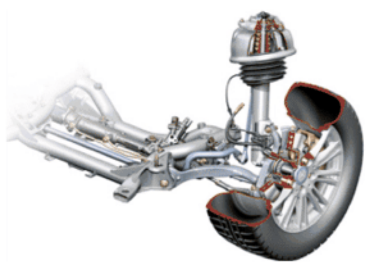
Identification theory

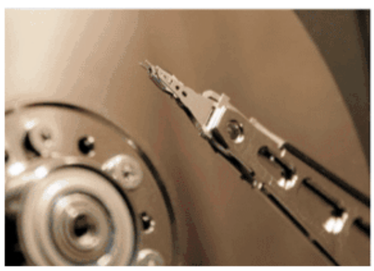
Identification Theory

Alexey Vedyakov

Application

- Measuring systems
- Disturbance compensation systems:
  - hard drives
  - ship
  - active suspension vehicle





First global convergent estimators

- Hsu L., Ortega R., Damm G. A globally convergent frequency estimator // IEEE Transactions on Automatic Control. 1999.
- G. Obregon-Pulido, B. Castillo-Toledo, A. A. Loukianov, "Globally convergent estimator for n-frequencies," IEEE Trans. Autom. Control, vol. 47, no. 5, pp. 857-863, May 2002.
- A. Bobtsov, A. Lyamin, D. Romasheva, "Algorithm of parameter's identification of polyharmonic function," in Proc. 15th IFAC World Congress on Automatic Control, Barcelona, Spain, Jul. 2002.
- X. Xia, "Global frequency estimation using adaptive identifiers," IEEE Trans. Autom. Control, vol. 47, no. 7, pp. 1188-1193, Jul. 2002.
- R. Marino, P. Tomei, "Global estimation of unknown frequencies," IEEE Trans. Autom. Control, vol. 47, no. 8, pp. 1324-1328, Aug. 2002.

Frequency estimation

Consider the measurable signal

$$y(t) = A\sin(\omega t + \phi), \tag{1}$$

where A is the amplitude, ω is the frequency, ϕ is the phase.

The goal is to obtain the frequency estimate $\hat{\omega}(t)$ such that

$$\lim_{t\to\infty} |\omega - \hat{\omega}(t)| = 0.$$

Sinusoidal signal generator

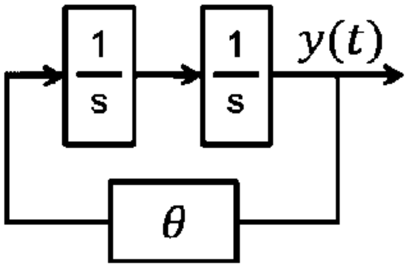
Consider derivatives of the signal (1)

$$\dot{y}(t) = \omega A \cos(\omega t + \phi),$$
$$\ddot{y}(t) = -\omega^2 A \sin(\omega t + \phi). \qquad (2)$$

Using (1) and (2) we can obtain linear regression model

$$\ddot{y}(t) = \theta y(t), \qquad (3)$$

where $\theta = -\omega^2$.



Gradient method

Consider the cost criterion

$$J(\theta) = \frac{1}{2}(\ddot{y}(t) - \hat{\ddot{y}}(t))^2 = \frac{1}{2}(\theta y(t) - \hat{\theta}(t)y(t))^2 = \frac{1}{2}e^2(t),$$

which we minimize with respect to $\hat{\theta}(t)$ using the gradient method

$$\dot{\hat{\theta}}(t) = -\gamma \nabla J(\hat{\theta}),$$

where $\gamma > 0$. In our case,

$$\nabla J(\hat{\theta}) = \frac{dJ}{d\theta} = -ey(t) = -y(t)\left(\ddot{y}(t) - \hat{\theta}(t)y(t)\right).$$

Finally,

$$\dot{\hat{\theta}}(t) = \gamma y(t)\left(\ddot{y}(t) - \hat{\theta}(t)y(t)\right).$$

Biased sinusoidal signal

Consider the measurable signal

$$y(t) = A_0 + A\sin(\omega t + \phi), \tag{4}$$

where A_0 is the constant bias. Consider derivatives of the signal (4)

$$\dot y(t) = \omega A\cos(\omega t + \phi), \tag{5}$$
$$\ddot y(t) = -\omega^2 A\sin(\omega t + \phi).$$
$$\dddot y(t) = -\omega^3 A\cos(\omega t + \phi). \tag{6}$$

Using (5) and (6) we can obtain linear regression model

$$\dddot y(t) = \theta\dot y(t).$$

The adaptive law

$$\dot{\hat\theta}(t) = \gamma\dot y(t)\left(\dddot y(t) - \hat\theta(t)\dot y(t)\right). \tag{7}$$

Modified version

Let us consider additional variable

$$\chi(t) = \hat\theta(t) - \gamma\dot y(t)\ddot y(t), \quad \text{then} \tag{8}$$
$$\hat\theta(t) = \chi(t) + \gamma\dot y(t)\ddot y(t). \tag{9}$$

Differentiating equation (9) we obtain

$$\dot{\hat\theta}(t) = \dot\chi(t) + \gamma\ddot y^2(t) + \gamma\dot y(t)\dddot y(t). \tag{10}$$

On the other hand, from (7) we have

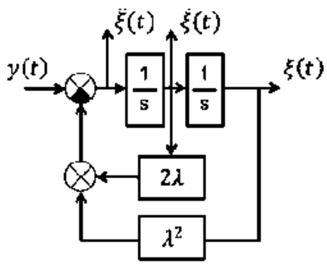
$$\dot{\hat\theta}(t) = \gamma\dot y(t)\left(\dddot y(t) - \hat\theta(t)\dot y(t)\right) = \gamma\dot y(t)\dddot y(t) - \gamma\hat\theta(t)\dot y^2(t). \tag{11}$$

Combining (10) and (11) gives

$$\hat\theta(t) = \chi(t) + \gamma\dot y(t)\ddot y(t),$$
$$\dot\chi(t) = -\gamma\hat\theta(t)\dot y^2(t) - \gamma\ddot y^2(t).$$

Without measuring derivatives

Let us consider linear filter



The signals ξ, $\dot{\xi}(t)$, $\ddot{\xi}(t)$ are measurable. Moreover,

$$\xi(t) = B_0 + B_1 \sin(\omega t + \psi) + \epsilon(t), \tag{12}$$

where $\epsilon(t)$ is exponentially decaying term. In this case,

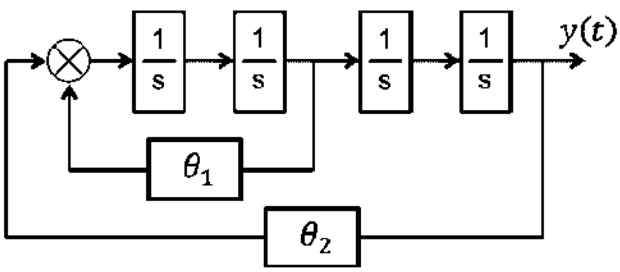
$$\begin{aligned}\hat{\theta}(t) &= \chi(t) + \gamma\dot{\xi}(t)\ddot{\xi}(t),\\ \dot{\chi}(t) &= -\gamma\hat{\theta}(t)\dot{\xi}^2(t) - \gamma\ddot{\xi}^2(t).\end{aligned}$$

Multi-Sinusoidal signal

Consider the measurable signal

$$y(t) = A_0 + \sum_{i=1}^{k} A_i \sin(\omega_i t + \phi_i). \tag{13}$$

Signal generator for $k = 2$



where $\theta_1 = -(\omega_1^2 + \omega_2^2)$, $\theta_2 = -\omega_1^2\omega_2^2$.

Multi-Sinusoidal signal

The signal (13) can be generated by the following differential equation

$$p(p^2-\theta_1)(p^2-\theta_2)\dots(p^2-\theta_k)y(t)=0, \qquad (14)$$

where $p = d/dt$ is the differentiation operator, $\theta_i = -\omega_i^2$, are constant parameters, $i=\overline{1,k}$. Equation (14) can be represented as

$$p^{2k+1}y(t)=\bar\theta_1 p^{2k-1}y(t)+\ldots+\bar\theta_k p y(t), \qquad (15)$$

where $\bar\theta_i$ can be calculated by the following system

$$\begin{cases} \bar\theta_1 = \theta_1+\theta_2+\dots+\theta_k; \\ \bar\theta_2 = -\theta_1\theta_2-\theta_1\theta_3-\dots-\theta_{k-1}\theta_k, \\ \vdots \\ \bar\theta_k = (-1)^{k+1}\theta_1\theta_2\cdots\theta_k. \end{cases}$$

General linear filter

Introduce the linear filter

$$\xi(s)=F(s)y(s), \quad F(s)=\frac{\lambda_0^{2k}}{\gamma(s)}, \qquad (16)$$

where $\lambda_0>0$, $\gamma(s)=s^{2k}+\gamma_{2k-1}s^{2k-1}+\dots+\gamma_1 s+\gamma_0$ is a Hurwitz polynomial.

Multiplying (15) by $\dfrac{\lambda_0^{2k}}{\gamma(s)}$ with (16) we obtain

$$s^{2k+1}\xi(s)=\bar\theta_1 s^{2k-1}\xi(s)+\ldots+\bar\theta_k s\xi(s).$$

Regression model

After the inverse Laplace transformation for the filter (16) and the input signal $y(t)$ we get the relation

$$\xi^{(2k+1)}(t) = \Omega^T(t)\bar{\Theta} + \varepsilon(t),$$

where $\Omega(t)$ is a regressor of functions $\xi^{(j)}(t)$

$$\Omega^T(t) = \begin{bmatrix}\xi^{(2k-1)}(t) & \dots & \xi^{(3)}(t) & \xi^{(1)}(t)\end{bmatrix},$$

$\bar{\Theta}$ is a vector of unknown parameters depending on frequencies

$$\bar{\Theta}^T = \begin{bmatrix}\bar{\theta}_1 & \dots & \bar{\theta}_{k-1} & \bar{\theta}_k\end{bmatrix}.$$

Adaptive Frequency Estimation

The update law

$$\hat{\omega}_i = \sqrt{\left|\hat{\theta}_i\right|}, \tag{17}$$

where estimates θ_i calculated using $\hat{\bar{\theta}}_i$ that are elements of a vector $\hat{\bar{\Theta}}$:

$$\hat{\bar{\Theta}} = \Upsilon(t) + K\Omega(t)\xi^{(2k)}(t), \tag{18}$$

$$\dot{\Upsilon}(t) = -K\Omega(t)\Omega^T(t)\hat{\bar{\Theta}}(t) - K\dot{\Omega}(t)\xi^{(2k)}(t). \tag{19}$$

where $K = \mathrm{diag}\{k_i > 0, i = \overline{1,k}\}$, guarantees that the estimation error $\tilde{\omega}_i = \omega_i - \hat{\omega}_i$ exponentially converges to zero:

$$|\tilde{\omega}_i(t)| \le \rho_1 e^{-\beta_1 t}, \quad \rho_1, \beta_1 > 0, \quad \forall t \ge 0. \tag{20}$$

Harmonics observer

For the variable $\xi(t)$ we have

$$\xi(t) = \xi_0 + \xi_1(t) + \xi_2(t) + \cdots + \xi_k(t). \tag{21}$$

After differentiation (21) $2k$ times, we obtain two systems of k linear equations:

$$\begin{cases} \xi^{(1)}(t) = \dot{\xi}_1(t) + \dot{\xi}_2(t) + \cdots + \dot{\xi}_k(t), \\ \xi^{(3)}(t) = \theta_1\dot{\xi}_1(t) + \theta_2\dot{\xi}_2(t) + \cdots + \theta_k\dot{\xi}_k(t), \\ \vdots \\ \xi^{(2k-1)}(t) = \theta_1^{k-1}\dot{\xi}_1(t) + \cdots + \theta_k^{k-1}\dot{\xi}_k(t), \end{cases}$$

and

$$\begin{cases} \xi^{(2)}(t) = \theta_1\xi_1(t) + \theta_2\xi_2(t) + \cdots + \theta_k\xi_k(t), \\ \xi^{(4)}(t) = \theta_1^2\xi_1(t) + \theta_2^2\xi_2(t) + \cdots + \theta_k^2\xi_k(t), \\ \vdots \\ \xi^{(2k)}(t) = \theta_1^k\xi_1(t) + \theta_2^k\xi_2(t) + \cdots + \theta_k^k\xi_k(t). \end{cases} \tag{22}$$

Harmonics observer

From (21) and (22) we get the realizable estimation scheme for variables ξ_0 and $\xi_i(t)$

$$\begin{bmatrix} \hat{\xi}_1(t) \\ \hat{\xi}_2(t) \\ \vdots \\ \hat{\xi}_k(t) \end{bmatrix} = \begin{bmatrix} \hat{\theta}_1 & \cdots & \hat{\theta}_k \\ \hat{\theta}_1^2 & \cdots & \hat{\theta}_k^2 \\ \vdots & \ddots & \vdots \\ \hat{\theta}_1^k & \cdots & \hat{\theta}_k^k \end{bmatrix}^{-1} \begin{bmatrix} \xi^{(2)}(t) \\ \xi^{(4)}(t) \\ \vdots \\ \xi^{(2k)}(t) \end{bmatrix},$$

and

$$\hat{\xi}_0 = \xi(t) - \sum_{i=1}^{k} \hat{\xi}_i(t).$$

Parameters estimation

The estimates of the amplitudes and phases

$$\hat{A}_i(t) = \frac{\hat{\gamma}_{\xi i}(t)}{\hat{L}_{\xi i}(t)}, \quad \hat{\phi}_i(t) = \left(-\hat{\varphi}_{\xi i}(t) + \hat{\phi}_{\xi i}(t)\right) \bmod 2\pi,$$

where

$$\hat{\gamma}_{\xi i}(t) = \sqrt{\hat{\xi}_i^2(t) + \left(\frac{\hat{\dot{\xi}}_i(t)}{\hat{\omega}_i(t)}\right)^2},$$

$$\hat{\phi}_{\xi i}(t) = \left(\operatorname{sign}\left(\hat{\xi}_i(t)\right)\arccos\left(\frac{\hat{\dot{\xi}}_i(t)}{\hat{\gamma}_{\xi i}(t)\hat{\omega}_i(t)}\right) - \hat{\omega}_i(t)t\right) \bmod 2\pi,$$

$\hat{L}_{\xi i}(t)$ and $\hat{\varphi}_{\xi i}(t)$ can be obtained from filter frequency response

$$\hat{L}_{\xi i}(t) = |F(j\omega)|_{\omega=\hat{\omega}_i}, \quad \hat{\varphi}_{\xi i}(t) = \arg F(j\omega)|_{\omega=\hat{\omega}_i}.$$

Dynamic Regressor Extension and Mixing

Consider the regression model

$$\psi(t) = \theta^T \varphi(t), \tag{23}$$

where $\psi(t) \in \mathbb{R}$ is the regressand, $\theta \in \mathbb{R}^n$ is the constant vector of unknown parameters, $\varphi(t) \in \mathbb{R}^n$ is the regressor.

Consider two linear operators

- The stable LTI filter. For example, we can choose exponentially stable LTI filters
$$H_l(p) = \frac{\lambda_l}{p + \lambda_l}, \tag{24}$$
where $p = \frac{d}{dt}$, $\lambda_l \in \mathbb{R}_+$, $l = \overline{1, n}$.
- The delay operator
$$[H_l(\cdot)](t) = (\cdot)(t - d_l), \tag{25}$$
where $d_l > 0$ is a delay.

Let us choose delay operator and define the filtered signals

$$\phi_{f,l}(t) = \phi(t - d_l), \tag{26}$$

$$\psi_{f,l}(t) = \psi(t - d_l). \tag{27}$$

Combine (26)–(27) and signals $\phi(t)$, $\psi(t)$ as follows

$$\Phi_e(t) = \begin{bmatrix} \phi^\top(t) \\ \phi_{f,1}^\top(t) \\ \vdots \\ \phi_{f,n-1}^\top(t) \end{bmatrix}, \quad \Psi_e(t) = \begin{bmatrix} \psi(t) \\ \psi_{f,1}(t) \\ \vdots \\ \psi_{f,n-1}(t) \end{bmatrix}, \tag{28}$$

where $\Phi(t) \in \mathbb{R}^{n\times n}$, $\Psi(t) \in \mathbb{R}^{n\times 1}$.

Defining

$$\zeta(t) = \det\{\Phi(t)\}, \tag{29}$$

$$\xi(t) = \mathrm{adj}\{\Phi(t)\}\Psi(t), \tag{30}$$

where $\det\{\Phi(t)\}$ is the determinant and $\mathrm{adj}\{\Phi(t)\}$ is the adjugate of matrix $\Phi(t)$, we obtain a set of n equations of the form

$$\xi_l(t) = \zeta(t)\theta_l, \quad l = \overline{1,n}. \tag{31}$$

In the obtained first order regression models (31) we can identify parameters θ_l separately.

The standard gradient method can be used for identification of the obtained models with scalar regressor and parameter

$$\dot{\hat{\theta}}_l(t) = \gamma_d \zeta(t) \left(\xi_l(t) - \zeta(t)\hat{\theta}_l(t) \right), \tag{32}$$

where $\gamma_d \in \mathbb{R}_+$.

From (31) and (32) we can write

$$\dot{\tilde{\theta}}_l(t) = -\gamma_d \zeta^2(t)\tilde{\theta}_l(t). \tag{33}$$

Solving this differential equation we obtain

$$\tilde{\theta}_l(t) = \tilde{\theta}_l(0) \exp\left(-\gamma_d \int_0^t \zeta^2(\tau)d\tau \right). \tag{34}$$

If $\zeta(t)$ is bounded and not square-integrable function, *i.e.*

$$\zeta(t) \notin \mathcal{L}^2 \leftrightarrow \int_0^\infty \zeta^2(\tau)d\tau = \infty, \tag{35}$$

then (32) provides convergence of the estimation error to zero, *i.e.*

$$\lim_{t\to\infty} \left\| \theta_l - \hat{\theta}_l(t) \right\| = 0. \tag{36}$$

For exponential convergence, the following inequality should hold

$$\int_0^t \zeta^2(\tau)d\tau \geq Dt, \tag{37}$$

where $D \in \mathbb{R}_+$.

Nonlinear control systems

Nonlinear Control Systems

Zimenko Konstantin

Nonlinear versus linear systems

Linear systems

- Huge body of work in analysis and control of linear systems
- Most models currently available are linear (but most real systems are nonlinear...)

Nonlinear systems

- Dynamics of linear systems are not rich enough to describe many commonly observed phenomena

Nonlinear systems can (sometime) be approximated by linear systems.
Nonlinear systems can (sometime) be "transformed" into linear systems.

2

State-space model

State equation

$$\dot{x} = f(t, x, u)$$

Output equation

$$y = h(t, x, u)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

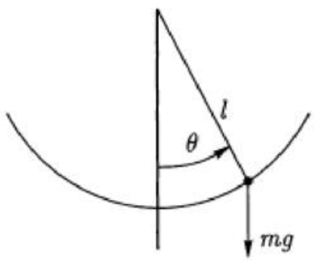
where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the input signal, and $y \in \mathbb{R}^q$ the output signal. The symbol $\dot{x} = \frac{dx}{dt}$ denotes the derivative of x with respect to time t.

3

Nonlinear systems: Example

Pendulum equation (equation of motion in the tangential direction)

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}.$$



State equations ($x_1 = \theta, x_2 = \dot{\theta}$)

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

Equilibrium points $(n\pi;0)$, $n = 0, \pm 1, \pm 2, \ldots$

4

Nonlinear systems: Example

State equations (frictional resistance is neglected)

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

Equilibrium points $(n\pi;0)$, $n = 0, \pm 1, \pm 2, \ldots$

State equations (with friction and applied torque)

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T$$

where T is the torque.

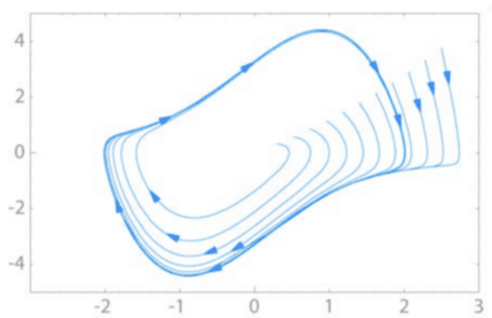
Equilibrium points $(\arcsin(T/mgl);0)$

5

Nonlinear systems: Example

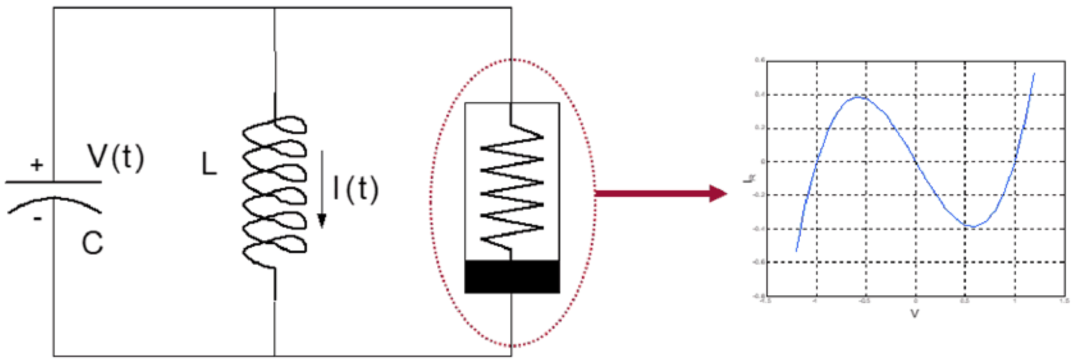
Robust oscillation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$



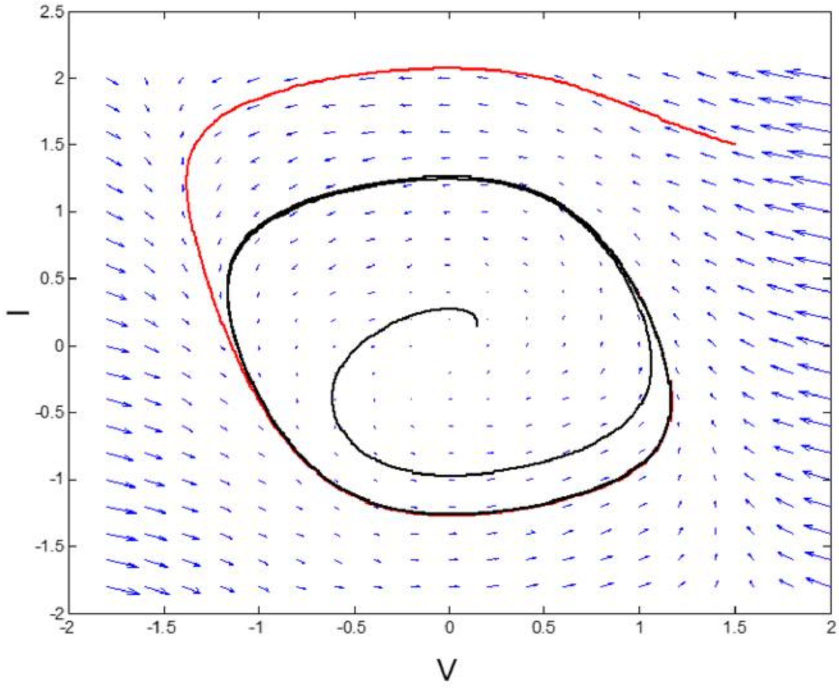
Van der Pol oscillator

$$L\dot{I} = V$$
$$C\dot{V} = -I - I_R(V)$$



Nonlinear systems: Example

Van der Pol oscillator: phase portrait



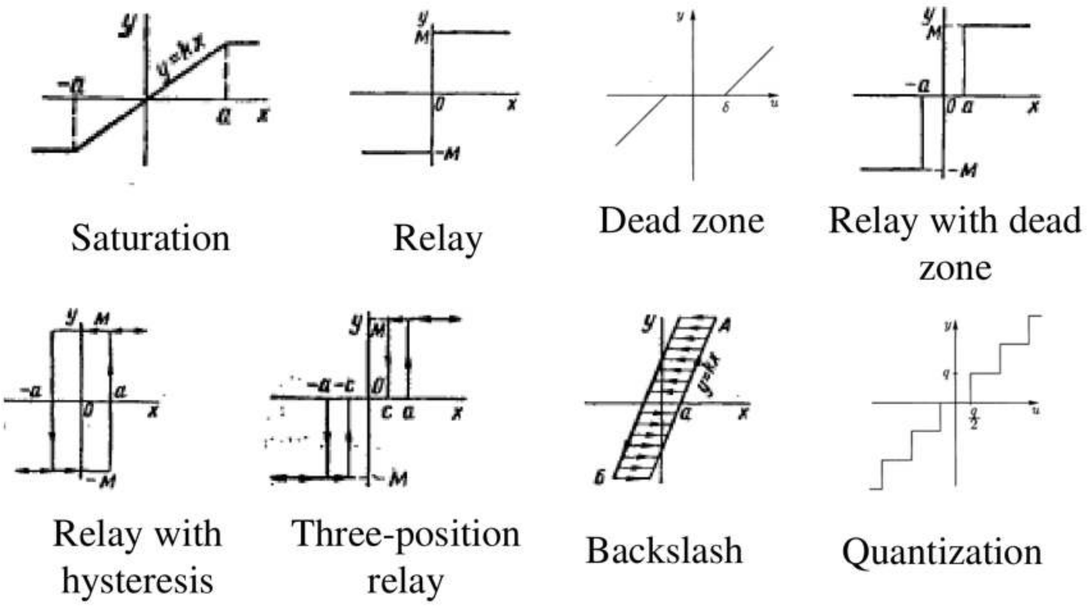
7

Nonlinear phenomena

- Finite escape time
  (the state of unstable linear system goes to infinity as $t \to \infty$)
- Nonasimptotic stability (e.g. finite-time stability)
  (linear systems – infinite time of convergence)
- Multiple isolated equilibria
  (linear systems – only one isolated equilibrium point)
- Limit cycles
  (linear systems – system oscilates iff there is a pair of eigenvalues on the imaginary axis, which is a nonrobust condition)
- Subharmonic, harmonic, or almost-periodic oscillations
  (stable linear system under periodic input produces an output of the same frequency)
- Chaos
  (More complicated steady-state behavior)
- Multiple modes of behavior

8

Common nonlinearities



9

Qualitative behavior of linear systems

Linear second order system

$$\dot{x} = Ax, \ \ x \in \mathbb{R}^2, A \in \mathbb{R}^{2\times 2}$$

Apply a similarity transformation M to A:

$$M^{-1}AM = J, \ \ M \in \mathbb{R}^{2\times 2}$$

where J is the real *Jordan form* of A, which depending on the eigenvalues of A may take one of the three forms

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

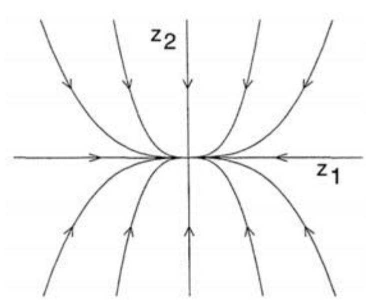
with k being either 0 or 1.

Present a change of coordinates

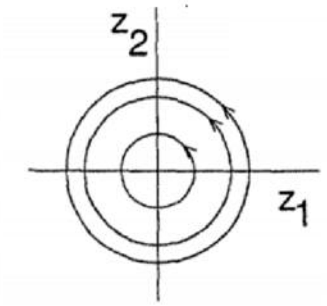
$$z = M^{-1}x$$
$$\dot{z} = M^{-1}\dot{x}$$

10

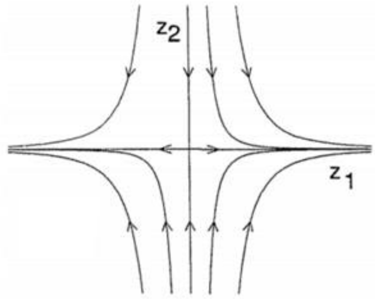
Qualitative behavior of linear systems



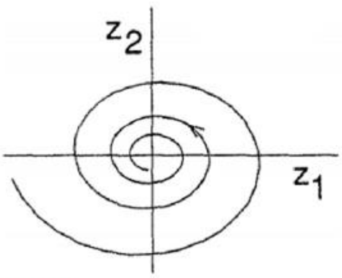
Stable node ($\lambda_{1,2} < 0$)



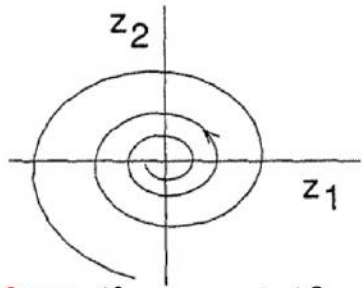
Center ($\lambda_{1,2} = \pm j\beta$)



Saddle point ($\lambda_2 < 0 < \lambda_1$)



Stable focus ($\lambda_{1,2} = a \pm j, a < 0$)

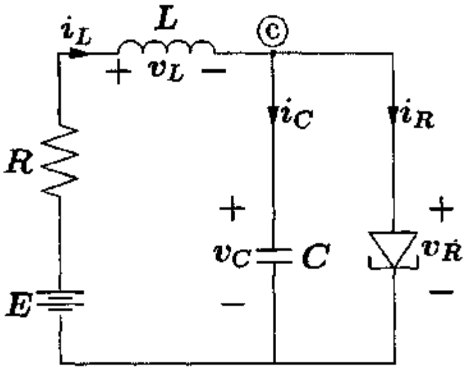


Stable focus ($\lambda_{1,2} = a \pm j\beta, a > 0$)

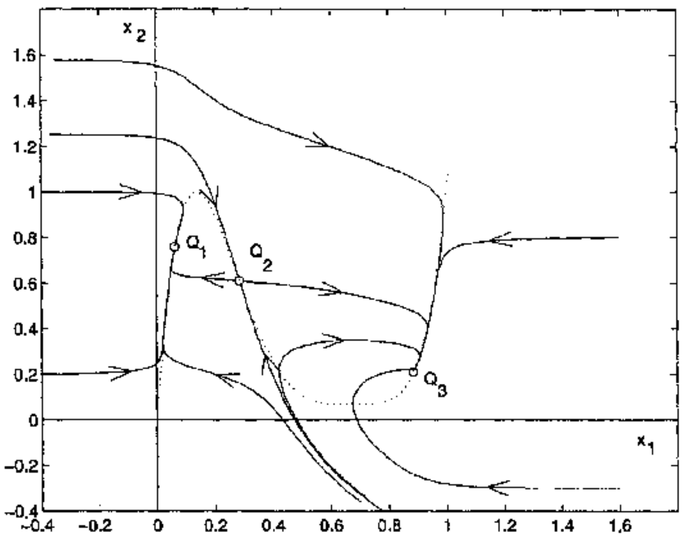
11

Multiple equilibrium points

Tunnel-diode circuit



$$\dot{x}_1 = \frac{1}{C}[-h(x_1) - x_2]$$
$$\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

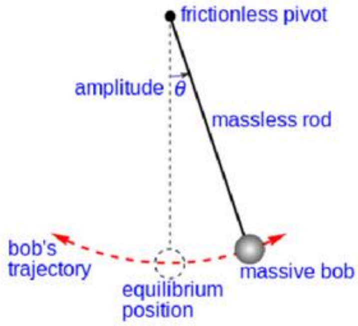


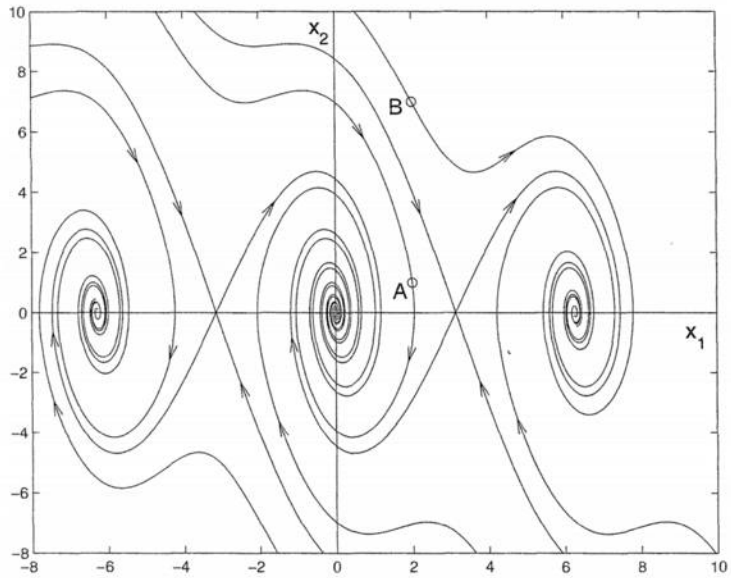
Phase portrait

12

Multiple equilibrium points

Pendulum





Phase portrait of the pendulum equation

13

Qualitative behavior near equilibrium

Consider autonomous system

$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ are continuously differeniable.

Let $p = (p_1, p_2)$ is the equilibrium point. Expanding the right-hand side into its Taylor series about the point p, obtain

$$\dot{x}_1 = f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + HOT,$$
$$\dot{x}_2 = f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + HOT,$$

where HOT denotes high order terms and

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x_1=p_1, x_2=p_2}, \quad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x_1=p_1, x_2=p_2}$$
$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x_1=p_1, x_2=p_2}, \quad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x_1=p_1, x_2=p_2}$$

Since $p = (p_1, p_2)$ is an equilibrium point

$$f_1(p_1, p_2) = f_2(p_1, p_2) = 0$$

14

Qualitative behavior near equilibrium

Define

$$y_1 = x_1 - p_1, \text{ и } y_2 = x_2 - p_2$$

and rewrite the state equation as

$$\dot{y}_1 = \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + HOT$$
$$\dot{y}_2 = \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + HOT$$

HOT is negligible in a small neighborhood of equilibrium point:

$$\dot{y}_1 = \dot{x}_1 = a_{11}y_1 + a_{12}y_2$$
$$\dot{y}_2 = \dot{x}_2 = a_{21}y_1 + a_{22}y_2$$

Rewriting in a vector form, obtain

$$\dot{y} = Ay$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=p} = \frac{\partial f}{\partial x}\Bigg|_{x=p}$$

15

Example 1

Pendulum equation

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -10\sin x_1 - x_2.$$

Equilibrium points (0;0) и (π;0)

Jacobian

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}$$

Jacobian evaluated at the equilibrium point

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \qquad \lambda_{1,2} = -0.5 \pm j3/12$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \qquad \lambda_{1,2} = -3.7, 2.7$$

Equilibrium point (0;0) is a stable focus, equilibrium point (π;0) is a saddle point

16

Example 2

Consider the system

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2)$$

Jacobian at (0;0) has eigenvalues $\pm j$.

Transition to polar coordinates:

$$x_1 = r\cos\theta \text{ и } x_2 = r\sin\theta$$

The system in polar coordinates

$$\dot{r} = -\mu r^3 \text{ и } \dot{\theta} = 1$$

For $\mu > 0$ the equilibrium point (0;0) is a stable focus, for $\mu < 0$ is a unstable focus.

17

Lyapunov function

Consider the system

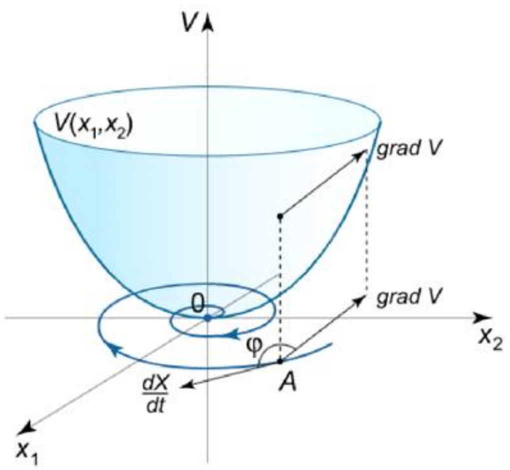
$$\dot{x} = f(x),$$

where $f: D \to R^n$ is locally Lipschitz

Let $p = 0$ is equilibrium point and $D \subset R^n$ is an open set, which contains p. Let $V : D \to R$ is a continuously differentiable function such, that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for } D \setminus \{0\}$$

If $\dot{V}(x) \le 0$ for $x \in D$, then $p = 0$ is stable.



If $\dot{V}(x) < 0$ for $x \in D \setminus \{0\}$, then $p = 0$ is asymptotically stable.

18

Lyapunov function

Consider the system

$$\dot{x} = f(x), f(0) = 0.$$

Expanding the right-hand side into its Taylor series

$$\dot{x} = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + g(x) = Ax + g(x),$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}.$$

Choose the candidate Lyapunov function in the form

$$V(x) = x^T P x, \quad P > 0$$

Then

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = \left[x^T A^T + g^T(x)\right] P x + x^T P[Ax + g(x)] = x^T \left(A^T P + PA\right) x + 2x^T P g(x) =$$
$$= -x^T Q x + 2x^T P g(x),$$

where $Q>0$ such, that

$$A^T P + PA = -Q \qquad \text{Lyapunov equation}$$

19

Lyapunov function

Let

$$|g(x)| < \gamma |x|,$$

where $\gamma > 0$.

Since

$$x^T Q x \ge \lambda_{\min}(Q) x^T x = \lambda_{\min}(Q) |x|^2,$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of the matrix Q, then

$$\dot{V}(x) \le -\lambda_{\min}(Q)|x|^2 + 2\gamma\|P\||x|^2 = -[\lambda_{\min}(Q) - 2\gamma\|P\|]x^2.$$

Lyapunov function derivative is negative if

$$\lambda_{\min}(Q) - 2\gamma\|P\| > 0 \Leftrightarrow \gamma < \frac{\lambda_{\min}(Q)}{2\|P\|}.$$

20

Example

Consider the system

$$\dot{x} = ax^3$$

Linearization:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.3ax^2\right|_{x=0} = 0.$$

Choose the Lyapunov function

$$V(x) = x^2.$$

Then

$$\dot{V}(x) = 2ax^4.$$

The equilibrium point is:

1) stable, if $a = 0$;
2) asymptotically stable, if $a < 0$;
3) unstable, if $a > 0$.

21

Stabilization: steady-state control

Consider the system

$$\dot{x} = f(x, u)$$

with desired equilibrium point $x = x_{\text{ss}}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{\text{ss}}, u_{\text{ss}})$$

$$x_\delta = x - x_{\text{ss}}, \qquad u_\delta = u - u_{\text{ss}}$$

$$\dot{x}_\delta = f(x_{\text{ss}} + x_\delta, u_{\text{ss}} + u_\delta) \overset{\text{def}}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \gamma(x_\delta) \quad \Rightarrow \quad u = u_{\text{ss}} + \gamma(x - x_{\text{ss}})$$

22

State feedback stabilization

Nonlinear system

$$\dot{x} = f(x, u) \qquad [f(0,0) = 0]$$
$$u = \gamma(x) \qquad [\gamma(0) = 0]$$

Problem: stabilize the system at the origin

$$\dot{x} = f(x, \gamma(x))$$

where f and γ are locally Lipschitz functions

23

Stabilization: linearization approach

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x,u)\bigg|_{x=0,u=0}; \quad B = \frac{\partial f}{\partial u}(x,u)\bigg|_{x=0,u=0}$$

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

$$\begin{aligned} \dot{x} &= \left[\frac{\partial f}{\partial x}(x,-Kx) + \frac{\partial f}{\partial u}(x,-Kx)\ (-K)\right]_{x=0} x \\ &= (A - BK)x \end{aligned}$$

$(A - BK)$ is Hurwitz $\Rightarrow$ the origin is an exponentially stable equilibrium point

24

Example: pendulum equation

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$$

Stabilize the pendulum at $\theta = \delta$

$$0 = -a\sin\delta + cT_{ss}$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ -a\cos(x_1 + \delta) & -b \end{array}\right]_{x_1=0} = \left[\begin{array}{cc} 0 & 1 \\ -a\cos\delta & -b \end{array}\right]$$

25

Example: pendulum equation

$$A = \left[\begin{array}{cc} 0 & 1 \\ -a\cos\delta & -b \end{array}\right]; \quad B = \left[\begin{array}{c} 0 \\ c \end{array}\right]$$

$$K = \left[\begin{array}{cc} k_1 & k_2 \end{array}\right]$$

$$A - BK = \left[\begin{array}{cc} 0 & 1 \\ -(a\cos\delta + ck_1) & -(b + ck_2) \end{array}\right]$$

$$k_1 > -\frac{a\cos\delta}{c}, \quad k_2 > -\frac{b}{c}$$

$$T = \frac{a\sin\delta}{c} - Kx = \frac{a\sin\delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}$$

26

Feedback linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{z} = Az + Bv$$

27

Feedback linearization

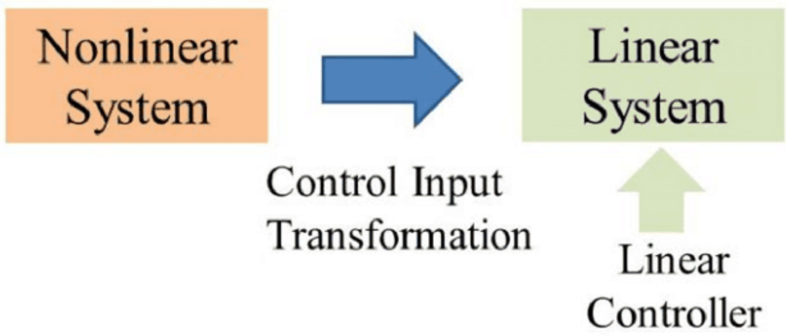
$$v = -Kz$$

Design K such that $(A - BK)$ is Hurwitz

$$u = \alpha(x) - \gamma^{-1}(x)KT(x)$$

Closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)\left[\alpha(x) - \gamma^{-1}(x)KT(x)\right]$$



28

Feedback linearization

Closed-loop system:

$$u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x) K \hat{T}(x)$$

$$\dot{z} = (A - BK)z + B\delta(z)$$

$$\delta = \gamma[\hat{\alpha} - \alpha + \gamma^{-1} KT - \hat{\gamma}^{-1} K \hat{T}]$$

where $\hat{\alpha}, \hat{\gamma}, \hat{T}$ are nominal models of α, γ and T.

$$V(z) = z^T P z, \quad P(A - BK) + (A - BK)^T P = -I$$

If $\|\delta(z)\| \le k\|z\|$ for all z, where

$$0 \le k < \frac{1}{2\|PB\|}$$

then the origin is globally exponentially stable

29

Example: pendulum equation

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss} = T - \frac{a}{c}\sin\delta$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

$$u = \frac{1}{c}\{a[\sin(x_1 + \delta) - \sin\delta] - k_1 x_1 - k_2 x_2\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix}$$ is Hurwitz

30

Example: pendulum equation

$$T = u + \frac{a}{c}\sin\delta = \frac{1}{c}\left[a\sin(x_1+\delta) - k_1x_1 - k_2x_2\right]$$

Let $\hat{a}$ and $\hat{c}$ be nominal models of a and c

$$T = \frac{1}{\hat{c}}\left[\hat{a}\sin(x_1+\delta) - k_1x_1 - k_2x_2\right]$$

$$\dot{x} = (A - BK)x + B\delta(x)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{c-\hat{c}}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

31

Example: pendulum equation

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{c-\hat{c}}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

$$|\delta(x)| \le k\|x\| + \varepsilon$$

$$k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c-\hat{c}}{\hat{c}}\right|\sqrt{k_1^2 + k_2^2}, \quad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| |\sin\delta_1|$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

$$\sin\delta_1 = 0 \quad \Rightarrow \quad \varepsilon = 0$$

32

Backstepping

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\xi \\ \dot{\xi} &= u, \quad \eta \in R^n, \ \xi, \ u \in R\end{aligned}$$

Stabilize the origin using state feedback

View ξ as "virtual" control input to

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \le -W(\eta), \quad \forall\, \eta \in D$$

33

Backstepping

$$z = \xi - \phi(\eta)$$

$$\begin{aligned}\dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= u - \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]\end{aligned}$$

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] + v$$

$$\begin{aligned}\dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v\end{aligned}$$

34

Backstepping

$$V_c(\eta, \xi) = V(\eta) + \tfrac{1}{2}z^2$$

$$\begin{aligned} \dot{V}_c &= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta} g(\eta) z + zv \\ &\le -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + zv \end{aligned}$$

$$v = -\frac{\partial V}{\partial \eta} g(\eta) - kz, \quad k > 0$$

$$\dot{V}_c \le -W(\eta) - kz^2$$

35

Backstepping

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x) z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\ \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \\ &\vdots \\ \dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1}) z_k \\ \dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k) u \end{aligned}$$

$$g_i(x, z_1, \ldots, z_i) \ne 0 \quad \text{for } 1 \le i \le k$$

36

Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Rightarrow \quad \dot{x}_1 = -x_1 - x_1^3$$

$$V(x_1) = \tfrac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V} = -x_1^2 - x_1^4, \quad \forall\, x_1 \in R$$

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \end{aligned}$$

37

Example

$$V_c(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

$$\begin{aligned} \dot{V}_c &= x_1(-x_1 - x_1^3 + z_2) \\ &\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \end{aligned}$$

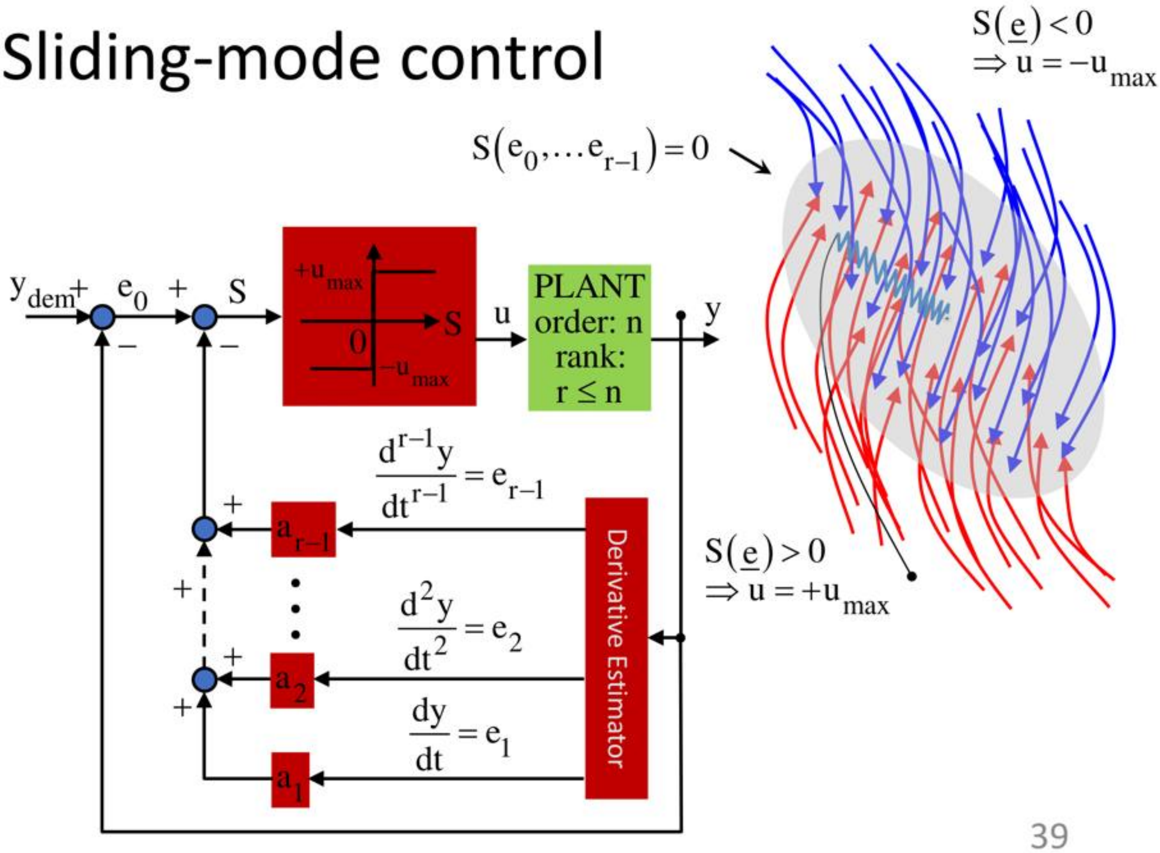
$$\begin{aligned} \dot{V}_c &= -x_1^2 - x_1^4 \\ &\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u] \end{aligned}$$

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2$$

Sliding-mode control



39

Sliding-mode control

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \quad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = a_1 x_1 + x_2 = 0$$

$$s(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = -a_1 x_1$$

$$a_1 > 0 \quad \Rightarrow \quad \lim_{t \to \infty} x_1(t) = 0$$

40

Sliding-mode control

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u$$

Suppose

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \le \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[a_1x_2 + h(x)] + g(x)su \le g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \ge \varrho(x) + \beta_0, \quad \beta_0 > 0$$

$$s > 0, \quad u = -\beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

41

Sliding-mode control

$$s < 0, \quad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

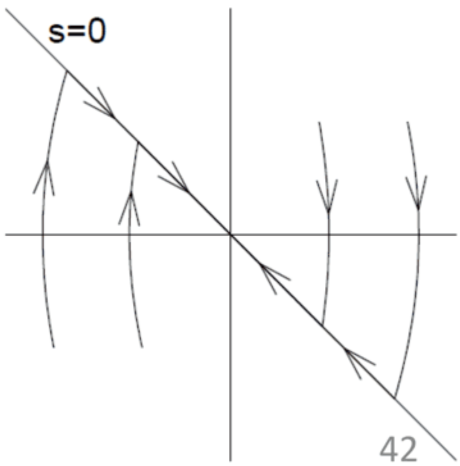
$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\,\mathrm{sgn}(s)$$

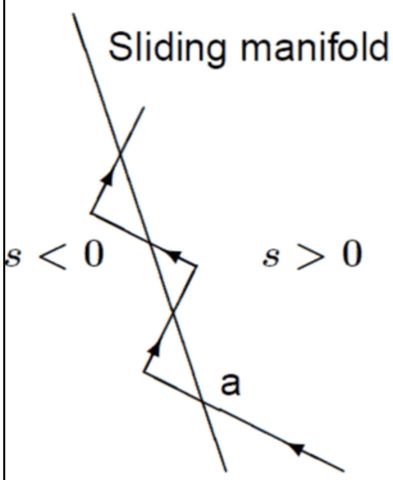
$$\dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$



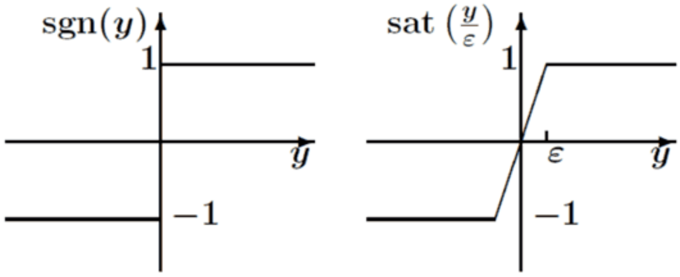
42

Sliding-mode control: chattering



$$u = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$



43

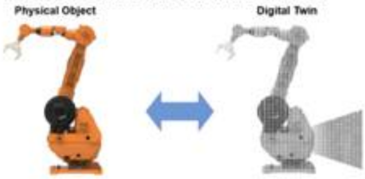
Digital twins

Cyber-physical systems Digital twins

Alexey Margun
alexeimargun@gmail.com

Digital twins

Digital Twin is a software analogue of a physical device that simulates internal processes, technical characteristics and behavior of a real object under the influence of the environment.



- Online copy of a real technical system (digital shadow)
- Offline modeling of technical systems

Digital twins

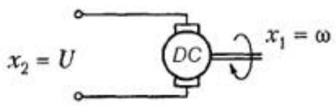
Problems:

- Unknown parameters
- Parameters changing
- Absence of sensors
- External noises and disturbances

Possible solutions:

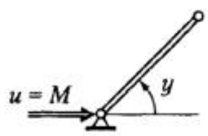
- Identification of unknown parameters
- Observers instead of sensors

Input-output model



$T\dot{x}_1(t) + x_1(t) = kx_2(t)$

$x_1(t) = k\left(1 - e^{-t/T}\right)x_2$



$J\ddot{y}(t) = u(t)$

Laplace transformation ($p = \frac{d}{dt}$): $\dot{x}_1 = px_1$, $\int x = \frac{x}{p}$.

Plant model:

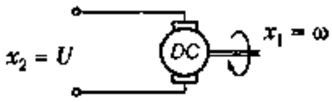
$$a(p)y(t) = b(p)u(t),$$

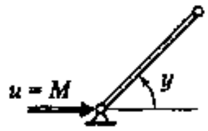
$a(p) = a_o p^n + a_1 p^{n-1} + \cdots + a_{n-1}p + a_n$ is a characteristic polynomial

$b(p) = b_o p^m + b_1 p^{m-1} + \cdots + b_{m-1}p + b_m$.

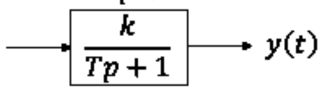
Input-output model

$y(t) = W(p)u(t), W(p) = \frac{b(p)}{a(p)}$ is a transfer function

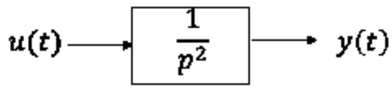




$$y(t) = \frac{k}{Tp+1}u(t)$$



$$y(t) = \frac{1}{p^2}u(t)$$



State space model

All linear differential equations ca be written as

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_1u, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_2u, \\ &\dots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_nu, \\ y(t) &= c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t). \end{aligned}$$

In matrix representation:

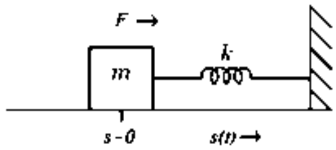
$$\dot{x} = Ax + Bu,$$
$$y = Cx$$

$$x = \begin{vmatrix} x_1 \\ \dots \\ x_n \end{vmatrix}, A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}, \mathrm{B} = \begin{vmatrix} b_1 \\ \dots \\ b_n \end{vmatrix}, C = [c_1 \quad \dots \quad c_n]$$

State space model. Example

Dynamic equations:

$$\dot{s} = v$$
$$m\dot{v} = F - ks - hv$$



State space model:

$$\dot{x} = Ax + Bu,$$
$$y = Cx$$

Let us choose state vector as $x = \begin{vmatrix} s \\ v \end{vmatrix}$.

$$A = \begin{vmatrix} 0 & 1 \\ -k/m & -h/m \end{vmatrix}, B = \begin{vmatrix} 0 \\ 1/m \end{vmatrix}, C = \begin{vmatrix} 1 & 0 \end{vmatrix}.$$

$$x(t) = x_{free} + x_{forced} = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

State space model. Change of coordinates

Consider new state vector:

$$x^* = Px$$

P is a transformation matrix, $\det P \neq 0$.

Inverse transformation:

$$x = P^{-1}x^*$$

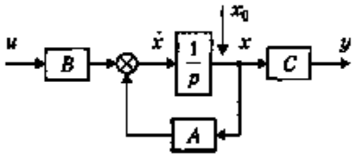
Model in new coordinates:

$$\begin{matrix} \dot{x}^* = A^*x^* + B^*u \\ y^* = C^*x^* \end{matrix}, A^* = PAP^{-1}, B^* = PB, C^* = CP^{-1}$$

Characteristic polynomial and poles of the system don't changes.

State space model

Modeling scheme



Transformation to input-output form:

$$W(p) = C(pI - A)^{-1}B,$$

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

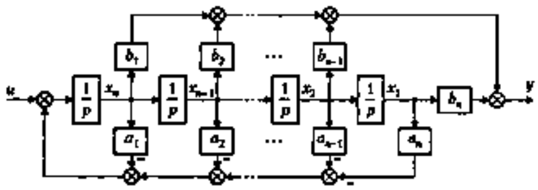
Characteristic polynomial:

$$\det(pI - A) = 0$$

State space model

Transformation to state space model:

$$W(p) = \frac{b_1p^{n-1} + \dots + b_{n-1}p + b_n}{p^n + a_1p^{n-1} + \dots + b_{n-1}p + b_n}$$



Canonical controlled form:

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

...

$\dot{x}_n = -a_nx_1 - a_{n-1}x_2 - \dots - a_1x_n + u$

$y = b_nx_1 + b_{n-1}x_2 + \dots + b_1$

$$A^* = \begin{vmatrix} 0 & | & I \\ & a^T & \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & & -a_1 \end{vmatrix}, B^* = \begin{vmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{vmatrix}, C^{*T} = \begin{vmatrix} b_n \\ b_{n-1} \\ \dots \\ b_2 \\ b_1 \end{vmatrix}$$

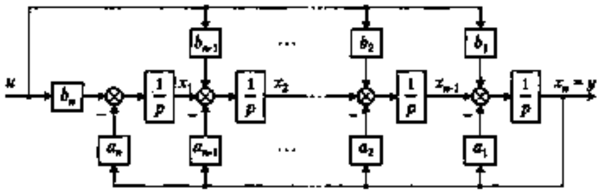
Transformation matrix: $P = U^*U^{-1}$, U, U^* are controllability matrices of canonical and original model

$$U = [B \,|\, AB \,|\, \dots \,|\, A^{n-1}B]$$

State space model

Transformation to state space model:

$$W(p) = \frac{b_1 p^{n-1} + \dots + b_{n-1}p + b_n}{p^n + a_1 p^{n-1} + \dots + b_{n-1}p + b_n}$$



Canonical observed form:

$\dot{x}_1 = -a_n x_n + b_n u$

$\dot{x}_2 = x_1 - a_{n-1} x_n + b_n u$

$\dot{x}_n = x_{n-1} - a_1 x_n + b_1 u$

$y = x_n$

$$A^* = \begin{vmatrix} 0^T | & \\ \dots | & a \\ I | & \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{vmatrix}, B^* = \begin{vmatrix} b_n \\ b_{n-1} \\ \dots \\ b_2 \\ b_1 \end{vmatrix}, C^{*T} = \begin{vmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{vmatrix}$$

Transformation matrix: $P = (Q^*)^{-1} Q$, Q, Q^* are observability matrices of canonical and original model

$$Q^T = [C | \, CA | \dots | CA^{n-1}]$$

Identification. Scalar example

Identification is a set of methods for constructing mathematical models of a dynamic systems from observational data.

Consider plant:

$$y(t) = \theta^* u(t)$$

$u(t)$ is a scalar input,
$y(t)$ is a scalar output,
θ^* is an unknown parameter.
The obvious solution :

$$\theta = \frac{y(t)}{u(t)}$$

Identification. Scalar example

Consider plant:

$$y(t) = \theta^* u(t)$$

$u(t)$ is a scalar input,
$y(t)$ is a scalar output,
θ^* is an unknown parameter.
The obvious solution :

$$\theta = \frac{y(t)}{u(t)}$$

Doesn't work if $u = 0$.
Hardly calculated if $u \to 0$.
High influence of noises.

Online estimation

Let θ is an estimate of θ^*.

Parallel model:

$$\hat{y}(t) = \theta u(t)$$

Error:

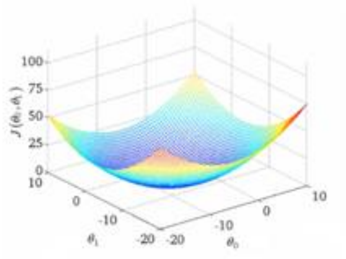
$$e = y - \hat{y} = y - \theta u$$

Consider functional:

$$J(\theta) = \frac{e^2}{2} = \frac{(y - \theta u)^2}{2}$$

Goal: minimize $J(\theta)$

Online estimation



Let denote:

$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \dots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$ is a gradient of $f(x)$.

Lemma. If $J \in C^1$ and is convex on R^n than θ^* is a global minimum if

$$\nabla J(\theta^*) = 0$$

Therefore, we need to solve equation $\nabla J(\theta^*) = 0$ with respect to the θ^*

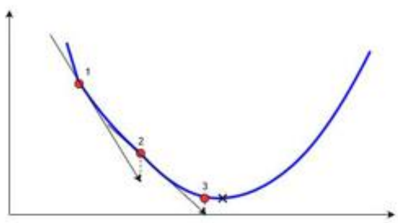
Gradient search. Discrete

The search for the minimum is in the direction of reducing the function $d_k = -\nabla J(\theta_k)$

Identification algorithm:

$$\theta_{k+1} = \theta_k + \lambda_k d_k = \theta_k - \lambda_k \nabla J(\theta_k),$$
$$k = 0, 1, 2, \dots$$

λ_k is a step size
θ_k is an estimate of θ on k-th step.

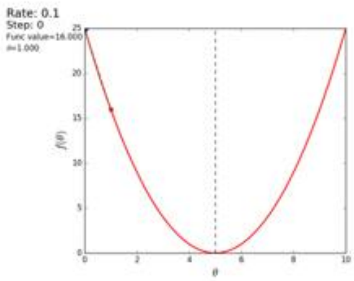


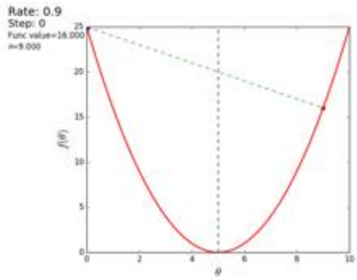
Gradient search. Discrete

Example:

$$y = (\theta - 5)^2$$
$$\frac{\partial y}{\partial \theta} = 2(\theta - 5)$$

Initial value $\theta = 0$.





Gradient search. Continuous

Rewrite algorithm as:

$$\frac{\theta_{k+1} - \theta_k}{\lambda_k} = -\nabla J(\theta)$$

If step is infinite small: $\lim\limits_{\lambda_k \to 0} \frac{\theta_{k+1} - \theta_k}{\lambda_k} = \dot{\theta}$

Algorithm takes the form:

$$\dot{\theta} = -\gamma \nabla J(\theta)$$

$\gamma > 0$ is a coefficient that regulates convergence speed

For scalar case

$$\dot{\theta} = -\gamma \nabla J(\theta) = \gamma(y - \theta u)u = \gamma e u, \theta(0) = \theta_0$$

Gradient search. Continuous

Consider estimation error:

$$\tilde{\theta} = \theta^* - \theta$$

Error transient:

$$\tilde{\theta}(t) = e^{-\gamma \int_0^t u^2(\tau)d\tau}\tilde{\theta}(0)$$

If $u = 0$ or $u = e^{-t}$, $\tilde{\theta}(t)$ will not converges to zero.

If $u^2 = \frac{1}{1+t}$, $\tilde{\theta}(t)$ asymptotically converges to zero.

$\tilde{\theta}(t)$ exponentially converges to zero if persistent excitation condition holds:

$$\int_t^{t+T_0} u^2(\tau)d\tau \geq \alpha_0 T_0, \forall t \geq 0$$

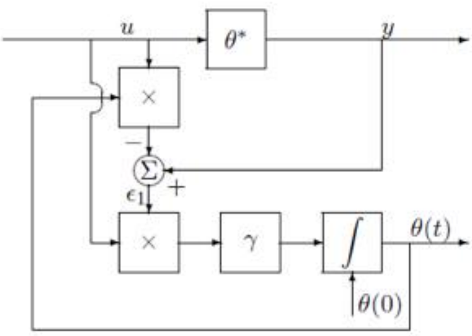
where $\alpha_0, T_0 > 0$

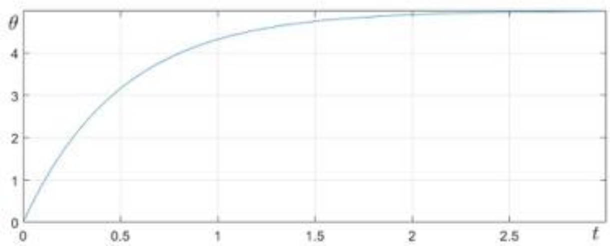
Gradient search. Example

Consider the system $y = \theta u, \theta = 5$

Identification algorithm: $\dot{\theta} = -\gamma \nabla J(\theta) = \gamma(y - \theta u)u = \gamma e u$

$\theta(0) = \theta_0, \gamma = 2$





Gradient search. Normalization

For system $y(t) = \theta^* u(t)$ with unbounded y and u problem

$$\min_\theta J = \min_\theta \frac{(y - \theta u)^2}{2}$$

can become hard for computing.

Solution is a normalization

$$\bar{y}(t) = \theta^* \bar{u}(t),$$

$$\bar{y}(t) = \frac{y}{m}, \bar{u}(t) = \frac{u}{m}, m^2 = 1 + u^2$$

Gradient search:

$$\dot{\theta} = \gamma \bar{e} \bar{u}, \gamma > 0.$$

In origin coordinates:

$$\dot{\theta} = \frac{\gamma e u}{m^2}$$

Gradient search. Two unknown

Consider system

$$\dot{x} = -ax + bu, x(0) = x_0,$$

$$\dot{x} = \theta^T \phi, \theta = [a \quad b]^T, \phi = [-x \quad u]$$

where $a > 0$ and b are unknown constants to be identified.

Parallel model:

$$\dot{\hat{x}} = -\hat{a}\hat{x} + \hat{b}\hat{u}, \hat{x}(0) = \hat{x}_0$$

Error:

$$e = x - \hat{x}$$

Functional:

$$J(\theta) = \frac{e^2}{2}$$

$$\dot{\theta} = \gamma \nabla J(\theta), \dot{\hat{a}} = -\gamma_1 e x, \dot{\hat{b}} = \gamma_2 e u$$

Gradient search. Two unknown

If $\dot{x}$ is unmeasured.

Rewrite system:

$$\dot{x} = -a_m x + (a_m - a)x + bu \text{ или } x = \frac{1}{p+a_m}[(a_m - a)x + bu]$$

$a_m > 0$ is chosen by developer.

$$x = \theta^{*T}\phi,$$

$$\theta^* = [b, a_m - a]^T, \phi = \left[\frac{1}{p+a_m}u, \frac{1}{p+a_m}x\right]^T$$

Error:

$$e = x - \hat{x}$$

Serial-parallel model:

$$\dot{\hat{x}} = -a_m\hat{x} + (a_m - \hat{a})x + \hat{b}u \text{ или } \hat{x} = \frac{1}{p+a_m}[(a_m - \hat{a})x + \hat{b}u]$$

$$J = \frac{e^2}{2}$$

$$\theta = [a \quad b]^T$$

$$\dot{\theta} = \gamma\nabla J(\theta), \dot{\hat{a}} = -\gamma_1 ex, \dot{\hat{b}} = \gamma_2 eu$$

Gradient search. Two unknown

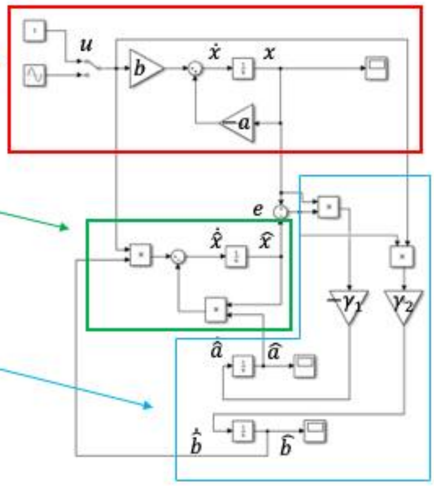
System: $\dot{x} = -ax + bu$.

Parallel model: $\dot{\hat{x}} = -a\hat{x} + \hat{b}u$.

Identification algorithm:

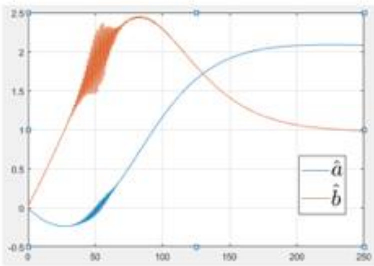
$\dot{\hat{a}} = -\gamma_1 ex$,

$\dot{\hat{b}} = \gamma_2 eu$

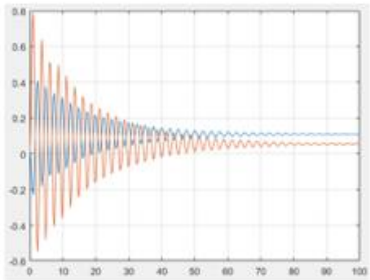


Gradient search. Two unknown

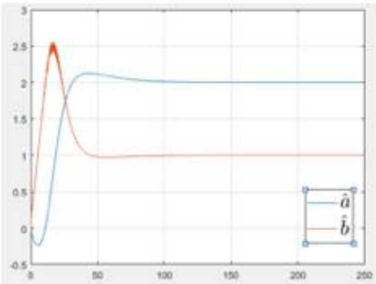
$u = \sin 5t$
$\gamma_{1,2} = 1$



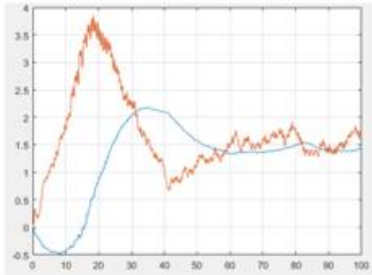
$u = 1(t)$
$\gamma_{1,2} = 5$



$\gamma_{1,2} = 5$



Белый шум



Gradient search. Linear dynamic system

Linear dynamic system:

$$\dot{x} = A \quad x + B \quad u$$
$$A \quad \in R^{nxn}, B \quad \in R^n, x \in R^n$$

Error:

$$e = x - \hat{x}$$

Functional:

$$J = \frac{e^T e}{2}$$

Parallel model:

$$\dot{\hat{x}} = \hat{A} \quad \hat{x} + \hat{B} \quad u, \hat{x} \in R^n$$
$$\dot{\hat{A}} = \gamma_1 e x^T, \dot{\hat{B}} = \gamma_2 e u^T$$

or serial-parallel model

$$\dot{x} = A_m x + (A \quad - A_m)x + B \quad u, A_m \in R^{nxn}$$
$$\dot{\hat{x}} = A_m \hat{x} + (\hat{A} \quad - A_m)\hat{x} + \hat{B} \quad u$$
$$\dot{\hat{A}} = \gamma_1 e x^T, \dot{\hat{B}} = \gamma_2 e u^T$$

System parametrization

Consider plant:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}u^{(n-2)} + \dots + b_0 u$$

Rewrite all parameters as vector

$$\theta^* = [b_{n-1}, b_{n-2}, \dots, b_0, a_{n-1}, a_{n-2}, \dots, a_0]^T$$

Rewrite input/output signals and their derivatives:

$$Y = \left[u^{(n-1)}, u^{(n-2)}, \dots, u, -y^{(n-1)}, -y^{(n-2)}, \dots, -y\right]^T =$$
$$= [\alpha_{n-1}^T(p)u, -\alpha_{n-1}^T(p)y]^T, \alpha_i(p) = \left[p^i, p^{i-1}, \dots, 1\right]^T$$

Therefore, we can rewrite system equation:

$$y^{(n)} = \theta^{*T} Y$$

System parametrization

If derivatives $y^{(n)} = \theta^{*T} Y$ are unmeasured

Apply stable filter $\frac{1}{\Lambda(p)}$ for both parts of equation, $\Lambda(p)$ is a Hurwitz polynomial:

$$z = \theta^{*T}\phi,$$
$$z = \frac{p^n}{\Lambda(p)}y, \phi = \left[\frac{\alpha_{n-1}^T(p)}{\Lambda(p)}u, -\frac{\alpha_{n-1}^T(p)}{\Lambda(p)}y\right]$$

$$\Lambda(p) = p^n + \lambda_{n-1}p^{n-1} + \dots + \lambda_0$$

All signals of filtered model are measured.

System parametrization

Consider $\Lambda(p)$ as $\Lambda(p) = p^n + \lambda^T \alpha_{n-1}(p), \lambda = [\lambda^{n-1}, \dots, \lambda_0]^T$
In this case:

$$z = \frac{p^n}{\Lambda(p)} y = \frac{\Lambda(p) - \lambda^T \alpha_{n-1}(p)}{\Lambda(p)} y = y - \lambda^T \frac{\alpha_{n-1}(p)}{\Lambda(p)} y$$

$$y = z + \lambda^T \frac{\alpha_{n-1}(p)}{\Lambda(p)} y$$

$$z = \theta^{*T}\phi = \theta_1^{*T}\phi_1 + \theta_2^{*T}\phi_2, \theta_1^{*T} = [b_{n-1}, \dots, b_0], \theta_2^{*T} = [a_{n-1}, \dots, a_0],$$

$$\phi_1 = \frac{\alpha_{n-1}(p)}{\Lambda(p)} u, \phi_2 = -\frac{\alpha_{n-1}(p)}{\Lambda(p)} \mathrm{y}$$

$$y = \theta_1^{*T}\phi_1 + \theta_2^{*T}\phi_2 - \lambda^T\phi_2$$

$$y = \theta_\lambda^{*T}\phi, \qquad \theta_\lambda^{*T} = [\theta_1^{*T}, \theta_2^{*T}, -\lambda^T]$$

State observers

Observer is an algorithm that allows to estimate the unmeasurable variables of the state vector.
Consider linear dynamic model:

$$\dot{x} = Ax + Bu,$$
$$y = C^T x$$

Parameters of system are known. Vector x is unmeasured.
If x_0 is known, than algorithm

$$\dot{\hat{x}} = A\hat{x} + Bu, \hat{x}(0) = x_0$$

provide $\hat{x}(t) = x(t) \forall t \geq 0$.

State observers

If x_0 is unknown and matrix A is stable we can use observer:

$$\dot{\hat{x}} = A\hat{x} + Bu, \hat{x}(0) = \hat{x}_0$$

Consider observation error:

$$\tilde{x} = x - \hat{x}$$

Its dynamics satisfy equation:

$$\dot{\tilde{x}} = A\tilde{x}, \tilde{x}(0) = x(0) - \hat{x}(0)$$

Solution of error dynamic equation:

$$\tilde{x}(t) = e^{At}\tilde{x}(0)$$

Because of A is stable $\tilde{x}$ exponentially converges to zero

Luenberger observer

If x_0 is unknown and matrix A is unstable or we need increase speed of convergence:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \hat{x}(0) = \hat{x}_0,$$
$$\hat{y} = C^T\hat{x},$$

where K is chosen by developer.

Dynamics of estimation error:

$$\dot{\tilde{x}} = (A - KC^T)\tilde{x}, \tilde{x}(0) = x(0) - \hat{x}(0)$$

So;ution of error dynamics equation:

$$\tilde{x}(t) = e^{(A-KC^T)t}\tilde{x}(0)$$

By tuning K we ensure the stability of the error model and adjust its transient (overshoot, transient time, etc.)

Luenberger observer. Example

System:

$$\dot{x} = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u, x(0) = \begin{bmatrix} 14 \\ 0{,}5 \end{bmatrix}$$
$$y = [1 \quad 0]x.$$

Luenberger observer

$$\dot{\hat{x}} = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} (y - \hat{y}),$$
$$\hat{y} = [1 \quad 0]\hat{x}.$$

Luenberger observer. Example

Denote $A_0 = A - KC^T = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -4 - k_1 & 1 \\ -4 - k_2 & 0 \end{bmatrix}$.

Let we need speed of convergence faster than e^{-5t}.

In this case real part of A_0 eigenvalues should be less than -5.
Let $\lambda_1 = -6, \lambda_2 = -8$.

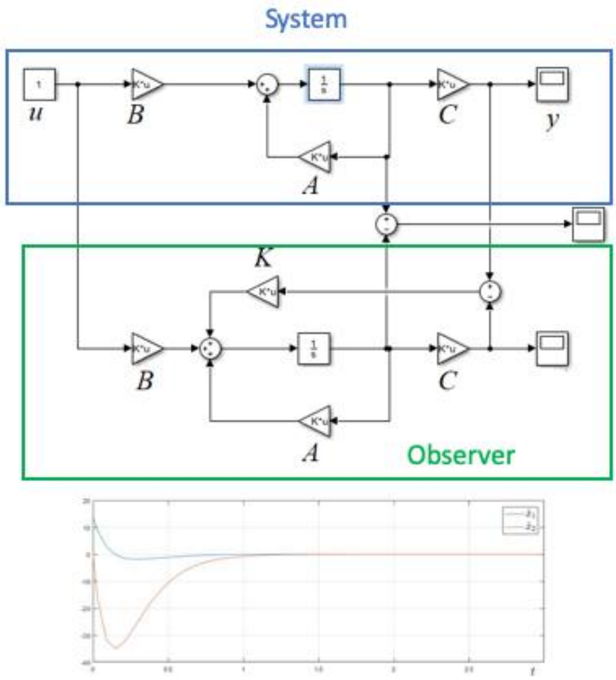
Therefore:

$$\det(pI - A_0) = p^2 + (4 + k_1)p + 4 + k_2 = (p + 6)(p + 8)$$

We can find:

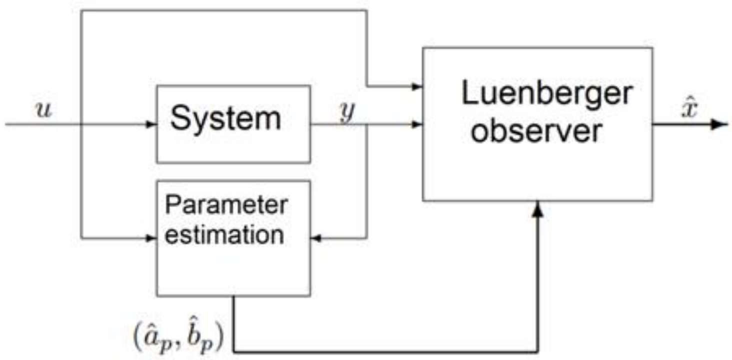
$$k_1 = 10, k_2 = 44$$

Luenberger observer. Example



Adaptive Luenberger observer

State vector is unmeasured.
System parameters are unknown.
Solution: simultaneously use the observer and the parameter estimation algorithm.



Adaptive Luenberger observer

In state space form we need to estimate $n^2 + 2n$ parameters.
In input output form we need to estimate $n + m + 1 \leq 2n$ parameters.
Obtain transfer function:

$$C^T(pI - A)^{-1}B = \frac{b_{n-1}p^{n-1} + \ldots + b_1 p + b_0}{p^n + a_{n-1}p^{n-1} + \cdots + a_0}$$

Rewrite system in canonical observable form:

$$\dot{x}_\alpha = \begin{bmatrix} & \vdots & I_{n-1} \\ -a_p & \vdots & \cdots \\ & \vdots & 0 \end{bmatrix} x_\alpha + b_p u, y = [1 \ 0 \ \ldots 0]x_\alpha$$

$$a_p = [a_{n-1}, a_{n-2}, \ldots, a_0]^T, b_p = [b_{n-1}, b_{n-2}, \ldots, b_0]^T$$

Adaptive Luenberger observer

Observer:

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}_p u + K(y - \hat{y}), \hat{x}(0) = \hat{x}_0,$$

$$\hat{y} = [1\ 0 \ldots 0]\hat{x},$$

$$\hat{A} = \begin{bmatrix} & \vdots & I_{n-1} \\ -\hat{a}_p & \vdots & \cdots \\ & \vdots & 0 \end{bmatrix}, K = a^* - \hat{a}_p$$

a^* is chosen such that

$$A^* = \begin{bmatrix} & \vdots & I_{n-1} \\ -a^* & \vdots & \cdots \\ & \vdots & 0 \end{bmatrix}$$

is stable, i.e. roots of $\det(pI - A^*) = 0$ have negative real part.

Digital twins

It is necessary for development of digital twin:

- Build mathematical model of system
- Estimate unknown parameters with identification algorithm
- Build observer for state vector estimation
- Run obtained model in real time with the same input signal as a real system

Modeling of systems and complexes

Modeling and control of robotic systems

Kinematics of industrial robots

Kinematics of Industrial Robots

Dr. Oleg Borisov

Basic Concepts and Definitions: Joints and Generalized Coordinates

> **Kinematic Chain**
> The *kinematic chain* is used to describe the geometry of the robot manipulator. Ot represents a graphic representation of the sequence of manipulator links connected by joints.

There are two elementary types of 1-DOF joints

- revolte (joint coordinate is angular)
- prismatic (joint coordinat is linear)

Both joint coordinates are so-called *generalized coordinates*

$$q_i = \begin{cases} \theta_i, & \text{if the link } i \text{ is revolute,} \\ d_i, & \text{if the link } i \text{ is prisnatic.} \end{cases} \tag{1}$$

> **Configuration**
> A set of all the generalized coordinates of the manipulator, which uniquely determines it in the space, is called *configuration.*

Basic Concepts and Definitions: FK and IK

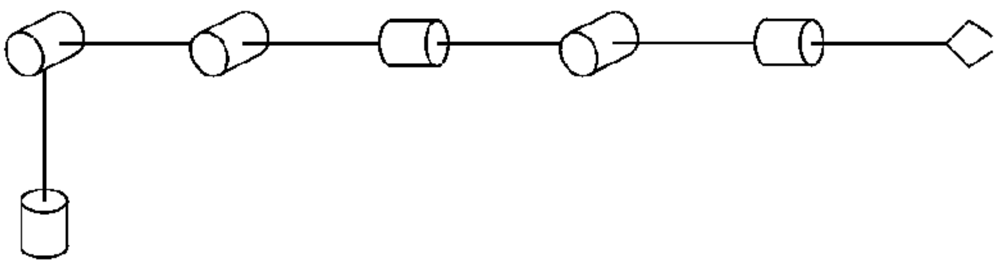
There are two fundamental tasks of the kinematics analysis

- forward kinematics
- inverse kinematics

> **Forward kinematics**
> The *forward kinematics (FK)* is to calculate the coordinates of the tool frame (its position and orientation) given the configuration of the robot.

> **Inverse kinematics**
> The *inverse kinematics (IK)* is to calculate the configuration of the robot given the coordinates of the tool frame (its position and orientation).

Forward Kinematics: Algorithm



Kinematic chain of 6-DOF robot

1. Assigning frames to the links.
2. Determining Denavit-Hartenberg parameters
3. Forming homogeneous transformation matrices
4. Parametrization of rotation matrix

Forward Kinematics: Assigning Frames

Choice of z_i-axes

Choose the axis z_i so that it coincides with the axis of rotation or translational motion of the subsequent joint $i+1$ depending on its type. This means that the relative location of adjacent links (coordinate systems) will be determined precisely by the variable around (or along) this axis.

Choice of x_i-axes

Choose the axis x_i, $i = \{1, 2 \ldots, n-1\}$ so that the following two conditions are satisfied.

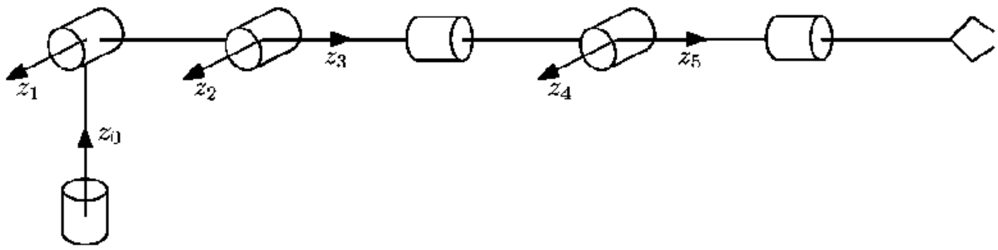
- The axis x_i is perpendicular to the axis z_{i-1}.
- The axis x_i intersects the axis z_{i-1}.

Choice of y_i-axes

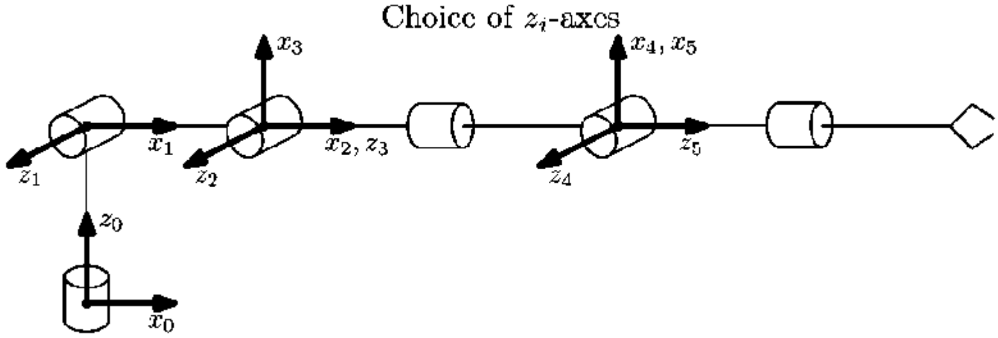
Choose the axis y_i so that the frame given by the unit vectors $\vec{x}_i$, $\vec{y}_i$, $\vec{z}_i$ is right-handed, i.e. in the direction given by the vector product:

$$\vec{y}_i = \vec{z}_i \times \vec{x}_i. \tag{2}$$

Forward Kinematics: Assigning Frames

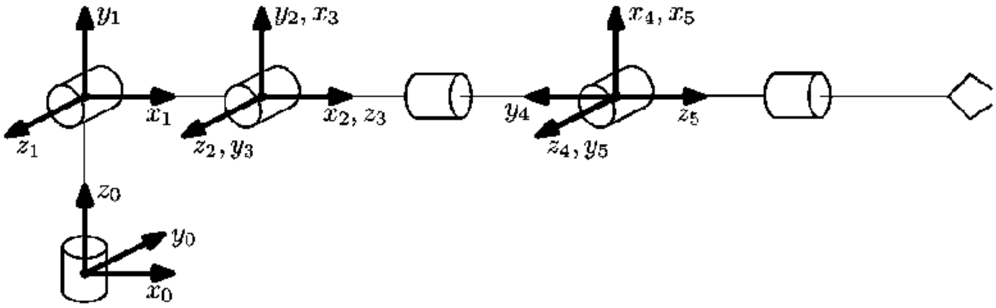


Choice of z_i-axes

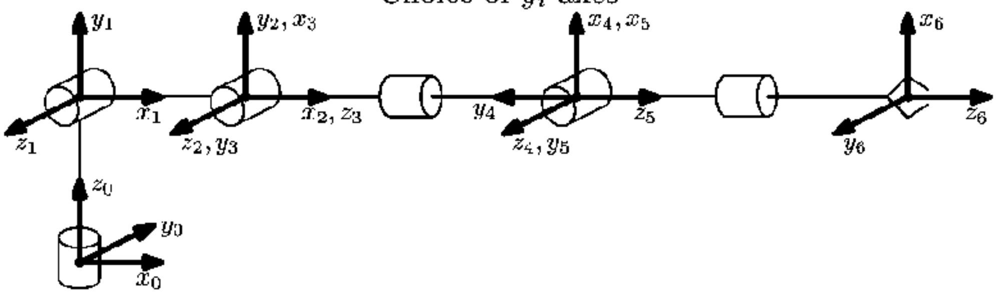


Choice of x_i-axes

Forward Kinematics: Assigning Frames



Choice of y_i-axes



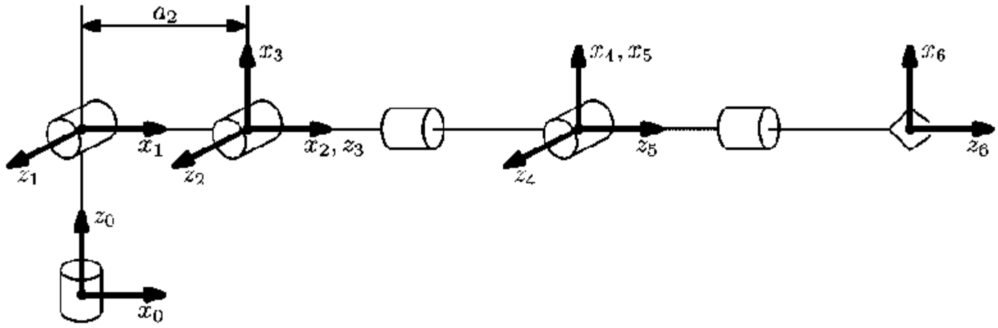
Choice of n-axes

Forward Kinematics: DH parameters

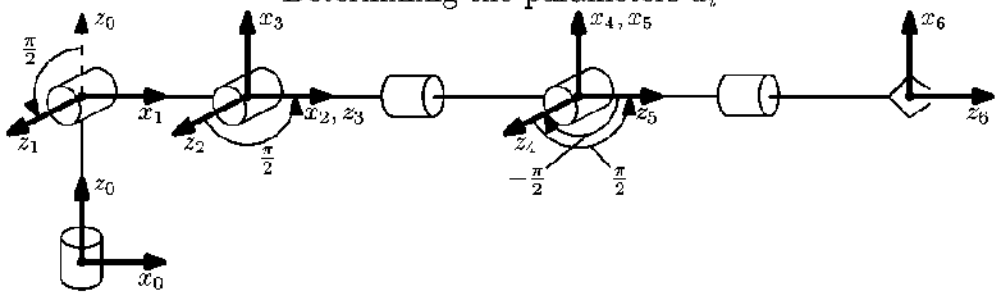
The Denavit-Hartenberg convention allows to reduce the number of coordinates that uniquely determine the body (its frame) in the space, from six to four, known as the *Denavite-Harteberg parameters* listed below.

- a_i is the distance along the axis x_i from z_{i-1} to z_i
- α_i is the angle around the axis x_i from z_{i-1} to z_i
- d_i is the distance along the axis z_{i-1} from x_{i-1} to x_i
- θ_i is the angle around the axis z_{i-1} from x_{i-1} to x_i

Forward Kinematics: DH parameters

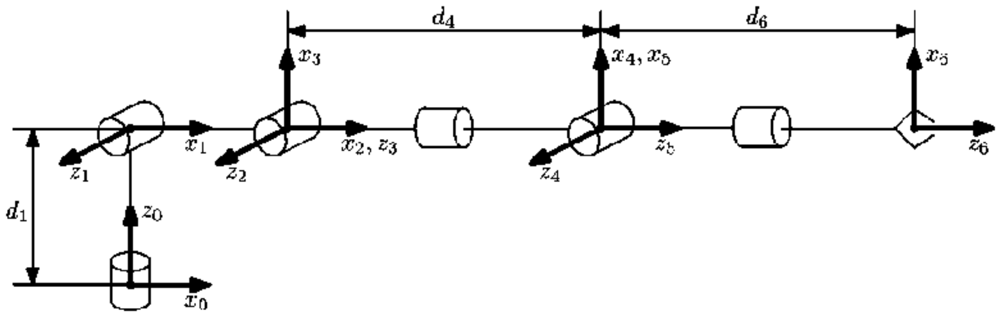


Determining the parameters a_i

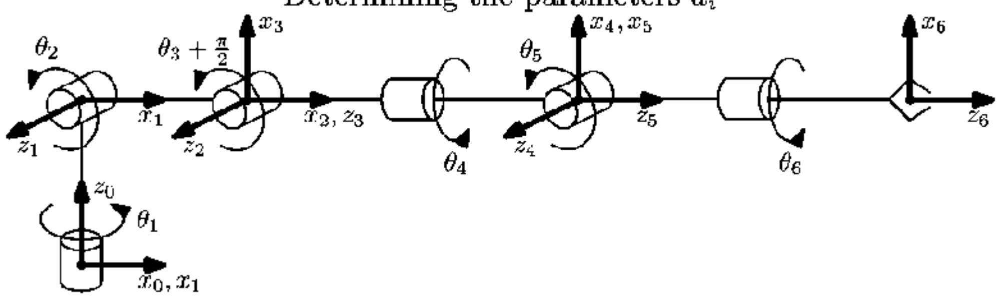


Determining the parameters α_i

Forward Kinematics: DH parameters



Determining the parameters d_i



Determining the parameters θ_i

Forward Kinematics: DH parameters

Link, i	a_i	α_i	d_i	θ_i
1	0	$\frac{\pi}{2}$	d_1	θ_1
2	a_2	0	0	θ_2
3	0	$\frac{\pi}{2}$	0	$\theta_3 + \frac{\pi}{2}$
4	0	$-\frac{\pi}{2}$	d_4	θ_4
5	0	$\frac{\pi}{2}$	0	θ_5
6	0	0	d_6	θ_6

DH parameters of the 6-DOF robot

Forward Kinematics: HT Matrix

Consider to sets of coordinates k^0 and k^n of the same point in the space expressed with respect to two frames $o_0x_0y_0z_0$ and $o_nx_ny_nz_n$, respectively:

$$k^0 = T_n^0 k^n, \tag{3}$$

where T_n^0 is the transformation carrying information about relative position and orientation of one frame with respect to another one.

Homogeneous Transformation Matrixe

The matrix T_n^0 defining the relation between frames $o_0x_0y_0z_0$ and $o_nx_ny_nz_n$ is called *a homogeneous transformation (HT) matrix* and has the form

$$T_n^0 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_n^0 & s_n^0 & a_n^0 & p_n^0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_n^0 & p_n^0 \\ 0 & 1 \end{bmatrix}, \tag{4}$$

where the vectors n_n^0, s_n^0 and a_n^0 express directions of x_n, y_n and z_n with respect to $o_0x_0y_0z_0$, $R_n^0 \in \mathcal{SO}(3)$ is the rotation matrix of the frame $o_nx_ny_nz_n$ with respect to $o_0x_0y_0z_0$, $p_n^0 \in \mathbb{R}^3$ is the vector of linear displacement of the origin of $o_nx_ny_nz_n$ with respect to $o_0x_0y_0z_0$.

Forward Kinematics: Properties of HT Matrix

1. The rotation by zero angle is determined by the identity matrix

$$R_{\beta=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \tag{5}$$

2. Rotation in the negative direction is determined by

$$R_{-\beta} = R_\beta^{-1} = R_\beta^T. \tag{6}$$

3. There are three basic rotation matrices around x, y and z axes given as

$$R_{x,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}, \; R_{y,\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \; R_{z,\beta} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where β is some angle.

Forward Kinematics: Properties of HT Matrix

4. Serial rotations around several *current* axes are determined by multiplying on the right. For example, the transformation parametrized by Euler angles ϕ, θ and ψ is given as

$$\begin{aligned} R_{zyz} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} = \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}, \end{aligned} \tag{7}$$

where $c_\beta \equiv \cos\beta$, $s_\beta \equiv \sin\beta$, $\beta = \{\phi, \theta, \psi\}$.

Forward Kinematics: Properties of HT Matrix

Using the DH convention form the homogeneous transformation matrices for each link as follows

$$T_i = T_{z,\theta_i} T_{z,d_i} T_{x,a_i} T_{x,\alpha_i} = \begin{bmatrix} R_{z,\theta_i} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p_{d_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p_{a_i} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,\alpha_i} & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (8)$$

where i is the link number, R_{z,θ_i} and R_{x,α_i} are the basic rotation matrices, p_{d_i} and p_{a_i} are vectors with nonzero components $p_z = d_i$ and $p_x = a_i$

$$R_{z,\theta_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{x,\alpha_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix},$$

$$p_{d_i} = \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix}, \qquad p_{a_i} = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix}. \qquad (9)$$

Forward Kinematics: Parametrization of Rotation Matrices

There different ways to parametrize rotation matrices

- Euler angles
- Roll-Pitch-Yaw angles
- Axis-Angle Representation

All of them are intended to reduce amount of parameters from 9 to 3.

Forward Kinematics: Euler Angles

The matrix of ZYZ-transformation is given as

$$R_n^0(q) = \begin{bmatrix} r_{11}(q) & r_{12}(q) & r_{13}(q) \\ r_{21}(q) & r_{22}(q) & r_{23}(q) \\ r_{31}(q) & r_{32}(q) & r_{33}(q) \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}. \quad (10)$$

Consider three cased depending on the entry $r_{33}(q)$.

First Case

If $r_{33}(q) \neq \pm 1$ then $\sin\theta(q) \neq 0$. Use the Pythagorean trigonometric identity

$$\sin^2\theta(q) + \cos^2\theta(q) = 1, \quad (11)$$

$$\sin(\theta(q)) = \pm\sqrt{1 - \cos^2\theta(q)} = \pm\sqrt{1 - r_{33}(q)} \quad , \quad (12)$$

from which it follows that $\theta(q)$ can be calculated as

$$\theta(q) = \operatorname{atan2}\left(\pm\sqrt{1 - r_{33}^2(q)}, r_{33}(q)\right). \quad (13)$$

Note that the remaining expressions to calculate $\phi(q)$ and $\psi(q)$ depend on the choice of the sign in front of the root in (13)

$$\phi(q) = \operatorname{atan2}(\pm r_{23}(q), \pm r_{13}(q)), \quad (14)$$

$$\psi(q) = \operatorname{atan2}(\pm r_{32}(q), \mp r_{31}(q)). \quad (15)$$

Second Case

If $r_{33}(q) = 1$ then $\cos\theta(q) = 1$, $\sin\theta(q) = 0$, from which $\theta(q) = 0$ and as a result

$$\begin{aligned} R_n^0(q) &= \begin{bmatrix} c_\phi c_\psi - s_\phi s_\psi & -c_\phi s_\psi - s_\phi c_\psi & 0 \\ s_\phi c_\psi + c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} c_{\phi+\psi} & -s_{\phi+\psi} & 0 \\ s_{\phi+\psi} & c_{\phi+\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\ &= \begin{bmatrix} r_{11}(q) & r_{12}(q) & 0 \\ r_{21}(q) & r_{22}(q) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{16}$$

This case leads to uncertainty, since only the sum $\phi(q) + \psi(q)$ can be computed

$$\phi(q) + \psi(q) = \text{atan2}\left(r_{21}(q), r_{11}(q)\right). \tag{17}$$

Third Case

If $r_{33}(q) = -1$ then $\cos\theta(q) = -1$, $\sin\theta(q) = 0$, from which $\theta(q) = \pi$, as a result

$$\begin{aligned} R_n^0(q) &= \begin{bmatrix} -c_\phi c_\psi - s_\phi s_\psi & c_\phi s_\psi - s_\phi c_\psi & 0 \\ -s_\phi c_\psi + c_\phi s_\psi & s_\phi s_\psi + c_\phi c_\psi & 0 \\ 0 & 0 & -1 \end{bmatrix} = \\ &= \begin{bmatrix} -c_{\phi-\psi} & -s_{\phi-\psi} & 0 \\ s_{\phi-\psi} & c_{\phi-\psi} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \\ &= \begin{bmatrix} r_{11}(q) & r_{12}(q) & 0 \\ r_{21}(q) & r_{22}(q) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{18}$$

This case leads to uncertainty, since only the difference $\phi(q) - \psi(q)$ can be computed

$$\phi(q) - \psi(q) = \text{atan2}\left(-r_{12}(q), -r_{11}(q)\right). \tag{19}$$

Inverse Kinematics

Initial data for IK are

- three linear coordinates (components of the vector p_n^0)
- three angular coordinates (e.g. Euler angles ϕ, ϕ and ψ)
- DH parameters

The geometric (analytical) method of solving IK is to find explicit expressions using the apparatus of trigonometric functions, taking into account the kinematic scheme of the manipulator.

Consider kinematic decoupling approach applied to standard 6-DOF robot with spherical wrist. It is comprised of two subtasks

- position IK (to compute q_1, q_2 and q_3)
- orientation IK (to compute q_4, q_5 and q_6)

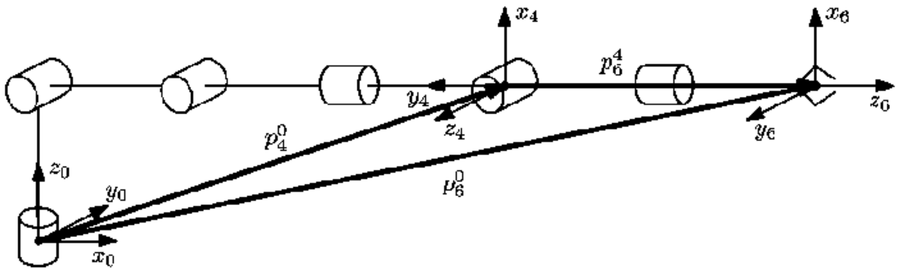
Inverse Kinematics: Position IK

> **Spherical wrist**
> A spherical wrist is a kinematic scheme of the last three rotational joints such that their axes of rotation intersect at the same point.

The subtask is

- to determine relations between the given point of the end-effector and the point of three axes intersection
- to derive expressions for q_1, q_2 and q_3 given the point of three axes intersection

Inverse Kinematics: Position IK



Kinematic decoupling

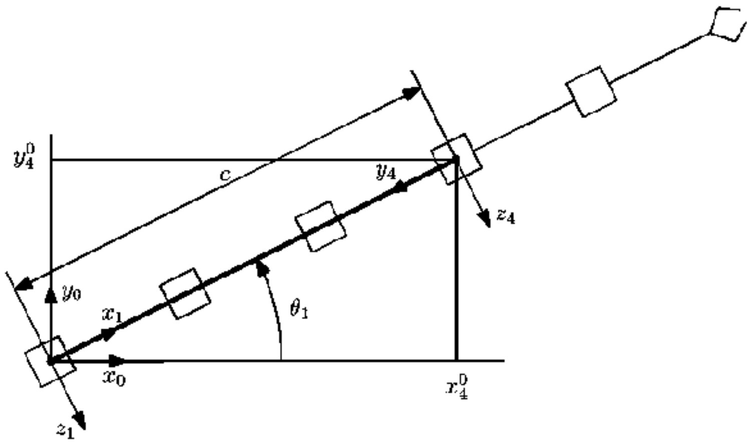
Using the sum of the vectors

$$p_6^0 = p_4^0 + d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{20}$$

express coordinates of the point as

$$p_4^0 = p_6^0 - d_6 R_6^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_4^0 \\ y_4^0 \\ z_4^0 \end{bmatrix}. \tag{21}$$

Inverse Kinematics: Position IK



Vector c

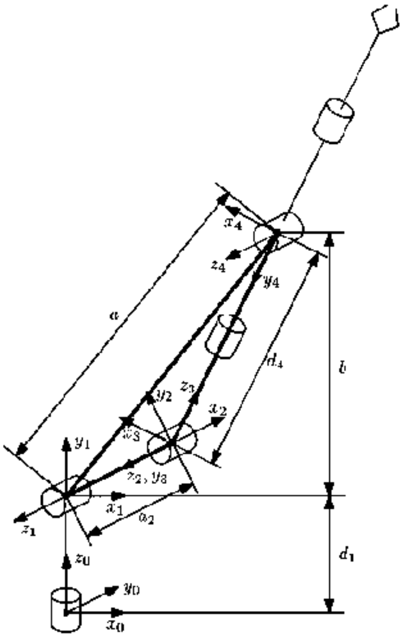
The first generalized coordinate can be computed as

$$\theta_1 = \text{atan2}(y_4^0, x_4^0) \tag{22}$$

or

$$\theta_1 - \text{atan2}(y_4^0, x_4^0) + \pi. \tag{23}$$

Inverse Kinematics: Position IK

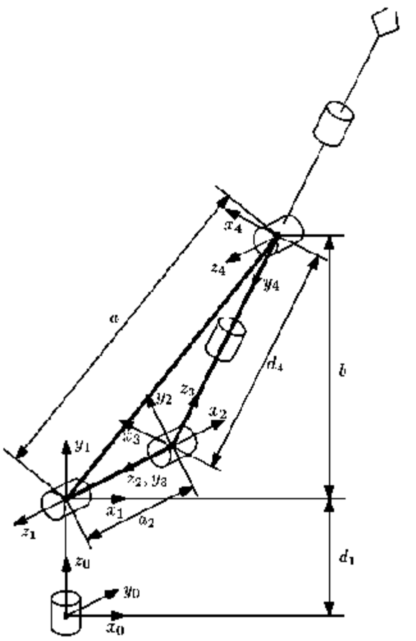


Vectors a and b

Use the following notations

$$
\begin{aligned}
a &= \sqrt{\left(x_4^1\right)^2 + \left(y_4^1\right)^2 + \left(z_4^1\right)^2}, && (24)\\
b &= \left(z_4^0 - d_1\right), && (25)\\
c &= \sqrt{\left(x_4^0\right)^2 + \left(y_4^0\right)^2}. && (26)
\end{aligned}
$$

Inverse Kinematics: Position IK



Vectors a and b

Using the Pythagorean theorem write

$$a^2 = b^2 + c^2. \quad (27)$$

Using the law of cosines write

$$
\begin{aligned}
a^2 &= a_2^2 + d_4^2 - 2a_2 d_4 \cos(\pi - \theta_3) = \\
&= a_2^2 + d_4^2 + 2a_2 d_4 \cos\theta_3. && (28)
\end{aligned}
$$

Combining the both expressions write

$$b^2 + c^2 = a_2^2 + d_4^2 + 2a_2 d_4 \cos\theta_3, \quad (29)$$

from which express $\cos\theta_3$

$$\cos\theta_3 = \frac{b^2 + c^2 - a_2^2 - d_4^2}{2a_2 d_4}. \quad (30)$$

As a result the generalized coordinate θ_3 can be computed as

$$\theta_3 = \operatorname{atan2}\left(\pm\sqrt{1 - \cos^2\theta_3}, \cos\theta_3\right). \quad (31)$$

Inverse Kinematics: Position IK

Consider difference between to angles

- angle α formed by a and c
- angle β formed by a and a_2

Express the generalized coordinate θ_2 as

$$\theta_2 = \alpha - \beta. \tag{32}$$

Taking into account trigonometric expressions

$$\tan\alpha = \frac{b}{c}, \tag{33}$$

$$\tan\beta = \frac{d_4 \sin\theta_3}{a_2 + d_4\cos\theta_3}, \tag{34}$$

rewrite (32) as

$$\theta_2 = \operatorname{atan2}(b, c) - \operatorname{atan2}(d_4 \sin\theta_3, a_2 + d_4\cos\theta_3). \tag{35}$$

Inverse Kinematics: Orientation IK

Express the rotation matrix R_6^0 as

$$R_6^0 = R_3^0 R_6^3, \tag{36}$$

where R_6^0 is given, R_3^0 can be calculated solving FK. Express R_6^3 as

$$R_6^3 = \left(R_3^0\right)^{-1} R_6^0 = \left(R_3^0\right)^T R_6^0. \tag{37}$$

Consider ZYZ-transformation given by the Euler angles as

$$R_6^3 = R_{zyz} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{38}$$

The remaining three generalized coordinates can be computed as

$$\theta_4 = \phi = \operatorname{atan2}\left(\pm r_{23}, \pm r_{13}\right), \tag{39}$$

$$\theta_5 = \theta = \operatorname{atan2}\left(\pm\sqrt{1 - r_{33}^2}, r_{33}\right), \tag{40}$$

$$\theta_6 = \psi = \operatorname{atan2}\left(\pm r_{32}, \mp r_{31}\right). \tag{41}$$

Inverse Kinematics: Summary

1. Solve forward kinematics
2. Calculate the coordinates of the intersection between the rotation axes given the coordinated of the tool
3. Solve position IK and get θ_1, θ_2 and θ_3
4. Calculate R_3^0 from forward kinematics
5. Calculate matrix R_6^3
6. Solve orientation IK and get θ_4, θ_5 and θ_6 as the Euler angles forming the matrix R_6^3

Dynamics of industrial robots

Dynamics of Industrial Robots

Dr. Oleg Borisov

Dynamical Model of Revolute Joint: Two Components

The electrical component of the model describes a circuit with the inductance, resistance and motor as

$$L\dot{i}(t) + Ri(t) = u(t) - K_\varepsilon\omega(t) = u(t) - K_\varepsilon\dot{\theta}(t), \tag{1}$$

where L, R, $i(t)$, $u(t)$ are the inductance, resistance, current and voltage of the armature, respectively, K_ε is the back emf constant, $\omega(t)$, $\theta(t)$ are the angular velocity and position of the rotor, respectively.

The mechanical component of the model describes a gear train with the gear ratio j connected with the motor as

$$J\ddot{\theta}(t) + K_f\dot{\theta}(t) = K_\mu i(t) - \mu_l(t), \tag{2}$$

where J is the sum of the actuator and gear moments of inertia, K_f is the friction constant, K_μ is the torque constant, $\mu_l(t) = \frac{1}{j}\mu_l(t)$, $\mu_l(t)$ is the load torque, j is the gear ratio.

Dynamical Model of Revolute Joint: Transfer Functions

Apply the Laplace transform and rewrite the model (1) and (2) as

$$\begin{aligned}(Ls + R)I(s) &= U(s) - K_\varepsilon s\Theta(s), &(3)\\ (Js + K_f)s\Theta(s) &= K_\mu I(s) - M_l(s). &(4)\end{aligned}$$

Taking into account (3) and (4) let us write the transfer function from the input $U(s)$ to the output $\Theta(s)$ with $M_l(s) = 0$

$$\frac{\Theta(s)}{U(s)} = \frac{K_\mu}{s((Ls + R)(Js + K_f) + K_\varepsilon K_\mu)}. \tag{5}$$

The transfer function from $M_l(s)$ to $\Theta(s)$ with $U(s) = 0$ is

$$\frac{\Theta(s)}{M_l(s)} = -\frac{Ls + R}{s((Ls + R)(Js + K_f) + K_\varepsilon K_\mu)}. \tag{6}$$

Dynamical Model of Revolute Joint: Simplification

Now divide numerator and denominator of the transfer functions (5) and (6) by R

$$\frac{\Theta(s)}{U(s)} = \frac{\frac{K_\mu}{R}}{s\left(\left(\frac{L}{R}s+1\right)(Js+K_f)+\frac{K_\varepsilon K_\mu}{R}\right)}, \tag{7}$$

$$\frac{\Theta(s)}{M_l(s)} = -\frac{\frac{L}{R}s+1}{s\left(\left(\frac{L}{R}s+1\right)(Js+K_f)+\frac{K_\epsilon K_\mu}{R}\right)}. \tag{8}$$

Since the time constant of the electrical component is reasonably much smaller than the time constant of the mechanical one

$$\frac{L}{R} << \frac{J}{K_f}, \tag{9}$$

rewrite transfer functions (7) and (8)

$$\frac{\Theta(s)}{U(s)} \approx \frac{\frac{K_\mu}{R}}{s\left(Js+K_f+\frac{K_\varepsilon K_\mu}{R}\right)}, \quad \frac{\Theta(s)}{M_l(s)} \approx \frac{-1}{s\left(Js+K_f+\frac{K_\epsilon K_\mu}{R}\right)}. \tag{10}$$

Dynamical Model of Revolute Joint: The Resultant Model

Define new notations for these transfer functions

$$\frac{\Theta(s)}{M_u(s)} \approx \frac{1}{s(Js+K)}, \tag{11}$$

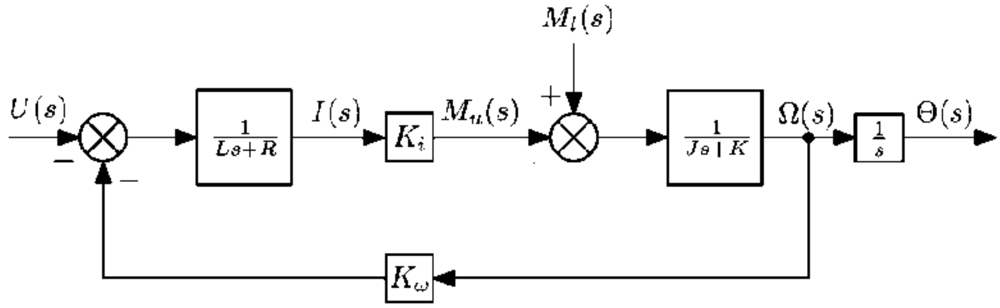
$$\frac{\Theta(s)}{M_l(s)} \approx \frac{1}{s(Js+K)}, \tag{12}$$

where $M_u(s) = \frac{K_\mu}{R}U(s)$, $K = K_f + \frac{K_\epsilon K_\mu}{R}$.

Combining transfer functions (11) and (12) we get

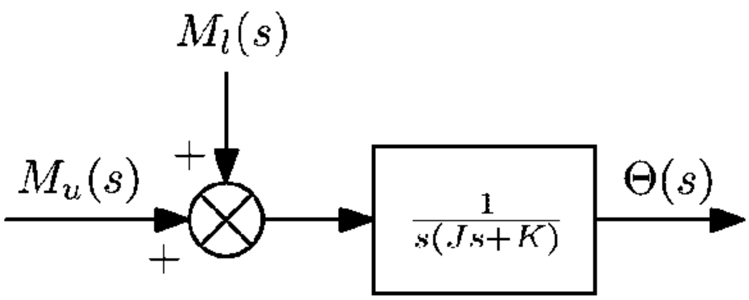
$$\Theta(s) = \frac{1}{s(Js+K)}\left(M_u(s) - M_l(s)\right) = P(s)\left(M_u(s) - M_l(s)\right). \tag{13}$$

Dynamical Model of Revolute Joint: Initial Scheme



Initial scheme of the revolute joint model

Dynamical Model of Revolute Joint: Simplified Scheme



Simplified scheme of the revolute joint model

Dynamical Model of the Robot: Euler-Lagrange Equation

Dynamics of mechanical systems can be described by the *Euler-Lagrange equation* as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \mu_i, \tag{14}$$

where L is the Lagrangian, q_i, $\dot{q}_i$ are the generalized coordinates and velocities, μ_i are the generalized torques applied to the joints.

The Langragian L can be computed as

$$L = K - P, \tag{15}$$

where K and L are the full kinetic and potential energies of the system, respectively.

Dynamical Model of the Robot: Kinetic Energy

The kinetic energy of the link is comprised of the linear and angular components

$$K_i = \frac{1}{2} m_i |v_i|^2 + \frac{1}{2} \omega_i^T I_i^0 \omega_i, \tag{16}$$

where m_i is the mass of the link, v_i is the linear velocity of the center of mass, ω_i is the angular velocity of the frame assigned with the link, I_i^0 is the inertia tensor with respect to the base frame.

Express the linear and angular velocities using the Jacobian matrix

$$v_i = J_{v_i}(q)\dot{q}, \tag{17}$$

$$\omega_i = J_{\omega_i}(q)\dot{q}. \tag{18}$$

Dynamical Model of the Robot: Kinetic Energy

Express the inertia tensor as

$$I_i^0 = R_i I R_i^T, \tag{19}$$

where R_i is the rotation matrix between the base and link frames, I is the inertia tensor with respect to the link frame given as $I = \begin{bmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ i_{31} & i_{32} & i_{33} \end{bmatrix}$,

where the elements are defined as

$$i_{11} = \iiint (y^2 + z^2)\rho(x, y, z)dxdydz, \quad i_{12} = i_{21} = -\iiint xy\rho(x, y, z)dxdydz,$$

$$i_{12} = \iiint (x^2 + z^2)\rho(x, y, z)dxdydz, \quad i_{13} = i_{31} = -\iiint xz\rho(x, y, z)dxdydz,$$

$$i_{13} = \iiint (x^2 + y^2)\rho(x, y, z)dxdydz, \quad i_{23} = i_{32} = -\iiint yz\rho(x, y, z)dxdydz,$$

where $\rho(x, y, z)$ is the funcition of mass density.

Rewrite the kinetic energy as

$$K_i = \frac{1}{2} m_i \dot{q}^T J_{v_i}^T J_{v_i} \dot{q} + \frac{1}{2} \dot{q}^T J_{\omega_i}^T R_i I R_i^T J_{\omega_i} \dot{q}. \tag{20}$$

Dynamical Model of the Robot: Full Energy

The full kinetic energy of the robot can be computed as

$$K = \frac{1}{2} \dot{q}^T \sum_{i=1}^{n} \left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T R_i I R_i^T J_{\omega_i} \right) \dot{q} = \frac{1}{2} \dot{q}^T \Lambda(q) \dot{q}. \tag{21}$$

The potential energy of the each link is computed as

$$P_i = m_i g^T p_i, \tag{22}$$

whre m_i is the mass of the link, g is the vector defining the direction of the gravitation with respect to the base frame, p_i is the radius-vector to the center of mass of the link expressed with respect to the base frame.

The full potential energy of the robot can be computed as

$$P = \sum_{i=1}^{n} m_i g^T p_i. \tag{23}$$

Dynamical Model of the Robot: Model of Multilink System

Substitute the kinetic and potential energies to the Lengrangian

$$L = \frac{1}{2}\dot{q}^T \Lambda(q)\dot{q} - \sum_{i=1}^{n} m_i g^T p_i. \tag{24}$$

Substitute the Langrangian to the Euler-Langrange equation

$$\Lambda(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \mu, \tag{25}$$

where $\Lambda(q) \in \mathbb{R}^{n\times n}$ is the symmetrical matrix of inertia, $C(q,\dot{q}) \in \mathbb{R}^{n\times 1}$ is the matrix of Coriolis forces, $G(q) \in \mathbb{R}^{n\times 1}$ is the vector of gravitational forces.

Dynamical Model of the Robot: Actuator Dynamics Revised

Write the dynamical model of the actuator dynamics as follows

$$J_i\ddot{\theta}_i(t) + F_i\dot{\theta}_i(t) = K_i\frac{u_i(t)}{r_i} - \mu_i(t), \tag{26}$$

where $F_i = K_f$, $K_i = K_\mu$, $r_i = R$, $\frac{u_i(t)}{r_i} = i(t)$, $i = \{1, 2, \dots n\}$ is the number of the link.

Take into account gear box

$$q_i = \frac{\theta_i}{j_i}. \tag{27}$$

Rewrite the actuator dynamics as

$$j_i^2 J_i\ddot{q}_i(t) + j_i^2 F_i\dot{q}_i(t) = j_i K_i\frac{u_i(t)}{r_i} - \bar{\mu}_i(t), \tag{28}$$

where $\mu_i = \mu_l$ for the link i.

Dynamical Model of the Robot: Actuator Dynamics Augmentation

Add the actuator dynamical model to the model of the mechanical system and get

$$\Gamma(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + G(q) = u, \tag{29}$$

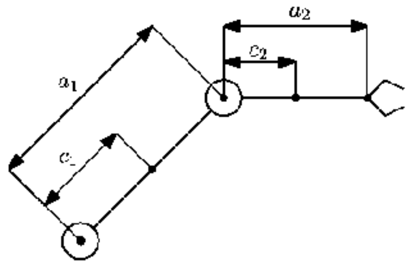
where the matrix $\Gamma(q)$ is of the form

$$\Gamma(q) = \Lambda(q) + J = \Lambda(q) + \begin{bmatrix} j_1^2 J_1 & 0 & \dots & 0 \\ 0 & j_2^2 J_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & j_n^2 J_n \end{bmatrix}, \tag{30}$$

where the friction vector and vector of control inputs are given respectively as

$$\begin{bmatrix} j_1^2 F_1 \\ j_2^2 F_2 \\ \vdots \\ j_n^2 F_n \end{bmatrix}, \quad \begin{bmatrix} j_1 K_1 \frac{u_1(t)}{R_1}(t) \\ j_2 K_2 \frac{u_2(t)}{R_2}(t) \\ \vdots \\ j_n K_n \frac{u_n(t)}{R_n}(t) \end{bmatrix}. \tag{31}$$

Example of Two-Link Planar Manipulator: Jacobian matrices



Kinematic chain of two-link robot

Write relations between linear end-effector velocities and generalized ones using the notion of the Jacobian matrix as follows

$$v_1 = J_{v,1}\dot{q}, \quad v_2 = J_{v,2}\dot{q}, \tag{32}$$

where

$$J_{v,1} = \begin{bmatrix} -a_1 \sin q_1 & 0 \\ c_1 \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}, J_{v,2} = \begin{bmatrix} -a_1 \sin q_1 - c_2 \sin(q_1+q_2) & -c_2 \sin(q_1+q_2) \\ a_1 \cos q_1 + c_2 \cos(q_1+q_2) & c_2 \cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}.$$

Example of Two-Link Planar Manipulator: Kinetic Energy

The kinetic energy is comprised of translational and rotational components. Let us address them separately. The translational component caused by the linear velocity can be computed as

$$K_{tr} = \frac{m_1 v_1^T v_1}{2} + \frac{m_2 v_2^T v_2}{2} = \underbrace{0.5\dot{q}^T m_1 J_{v,1}^T J_{v,1}\dot{q}}_{\text{1st link}} + \underbrace{0.5\dot{q}^T m_2 J_{v,2}^T J_{v,2}\dot{q}}_{\text{2nd link}} \tag{33}$$

The rotational component caused by the angular velocity can be computed as

$$K_{rt} = \underbrace{0.5\dot{q}^T I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{q}}_{\text{1st link}} + \underbrace{0.5\dot{q}^T I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dot{q}}_{\text{2nd link}} \tag{34}$$

Example of Two-Link Planar Manipulator: Inertia Matrix

The inertia matrix $\Lambda(q)$ becomes of the form

$$\Lambda(q) = m_1 J_{v,1}^T J_{v,1} + m_2 J_{v,2}^T J_{v,2} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 c_1^2 + m_2(a_1^2 + c_2^2 + 2a_1 c_2^2 + 2a_1 c_2 \cos q_2) + I_1 + I_2 & m_2(c_2^2 + a_1 c_2 \cos q_2) + I_2 \\ m_2(c_2^2 + a_1 c_2 \cos q_2) + I_2 & m_2 c_2^2 + I_2 \end{bmatrix}$$

Example of Two-Link Planar Manipulator: Matrix of Coriolis Forces

Each element of the matrix of Coriolis forces $C(q)$ can be calculated using the equation

$$c_{kj} = \sum_{i=1}^{n} 0.5 \left(\frac{\partial \lambda_{kj}}{\partial q_i} + \frac{\partial \lambda_{ki}}{\partial q_j} - \frac{\partial \lambda_{ij}}{\partial q_k} \right) \dot{q}_i \qquad (35)$$

The matrix of Coriolis forces $C(q)$ becomes of the form

$$C(q) = \begin{bmatrix} -m_2 a_1 c_2 \sin q_2 \dot{q}_2 & -m_2 a_1 c_2 \sin q_2 (\dot{q}_2 + \dot{q}_1) \\ m_2 a_1 c_2 \sin q_2 \dot{q}_1 & 0 \end{bmatrix} \qquad (36)$$

Example of Two-Link Planar Manipulator: Vector of Gravitational Forces

Each element of the vector of gravitational forces $G(q)$ can be calculated using the equation

$$g_i = \frac{\partial P}{\partial q_i} \qquad (37)$$

The vector of gravitational forces $G(q)$ becomes of the form

$$G(q) = \begin{bmatrix} (m_1 c_1 + m_2 a_1) g \cos q_1 + m_2 c_2 g \cos(q_1 + q_2) \\ m_2 c_2 \cos(q_1 + q_2) \end{bmatrix} \qquad (38)$$

Example of Two-Link Planar Manipulator: Resultant Model

The resultant model of the two-link robot is

$$\lambda_{11}\ddot{q}_1 + \lambda_{12}\ddot{q}_2 + c_{11}\dot{q}_1 + c_{12}\dot{q}_2 + (m_1c_1 + m_2a_1)g\cos q_1 + m_2c_2g\cos(q_1 + q_2) = \mu_1$$

$$\lambda_{21}\ddot{q}_1 - \lambda_{22}\ddot{q}_2 + c_{21}\dot{q}_1 + m_2c_2\cos(q_1 + q_2) = \mu_2$$

Summary

- Dynamical models of industrial robots allow to describe and take into account (designing a control law) physical processes specific to them
- The simplified model of the revolute joint can be represented by the transfer function of the relative degree 2
- The dynamical model of the industrial robot can be derived using the Euler-Lagrange approach

Motion planning for industrial robots

Motion Planning for Industrial Robots

Dr. Oleg Borisov

Basic Concepts and Definitions: Configuration Space

Configuration

A configuration q is a set of all intermediate generalized coordinates (joint variables).

Configuration space

Configirations space $\mathcal{Q}$ is a set of all possible configurations q

$$\mathcal{Q} = \{q\}. \tag{1}$$

Basic Concepts and Definitions: Workspace

Workspace

Workspace $\mathcal{W}$ is a set of points, which belong to the robot itself and the reachable environment including all the obstacles

$$\mathcal{R}(q) \subset \mathcal{W}, \quad \mathcal{O} \subset \mathcal{W}, \tag{2}$$

where $\mathcal{R}(q)$ is space occupied by the robot and $\mathcal{O}$ is space occupied by the obstacles.

In case of a planar manipulator which movements are constrained by the plane

$$\mathcal{W} \subset \mathbb{R}^2, \tag{3}$$

its workspace has two-dimensional.

In case of a spatial manipulator, which is able to move along three orthogonal axes

$$\mathcal{W} \subset \mathbb{R}^3, \tag{4}$$

its workspace is three-dimensional.

Basic Concepts and Definitions: Collision-Free Space

Collision-Free Space

Space corresponding to collision of the robot with some obstacle is defined as follows

$$\mathcal{Q}_\times = \{q \in \mathcal{Q} | \mathcal{R}(q) \cap \mathcal{O} \neq 0\}, \tag{5}$$

from which collision-free space can be expressed as

$$\mathcal{Q}_0 = \mathcal{Q} \setminus \mathcal{Q}_\times. \tag{6}$$

Basic Concepts and Definitions: Path and Trajectory

Path Planning
Path planning is a process of searching a cosecutive set of configurations within collision-free space connecting the initial configuration with the given final one.

Trajectory Planning
Trajectory planning is a process of time parametrization of the path, i.e. computation of reference functions of time for generalized coordinates, velocities and accelerations.

Path Planning: Exact Cell Decomposition Approach

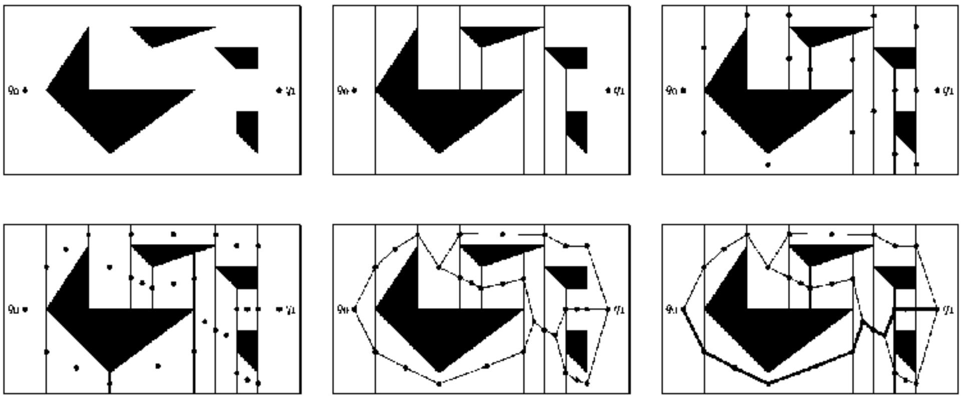
Exact Cell Decomposition
The idea of *exact cell decomposition* is to divide whole free configuration space on triangle or trapezoid cells and to construct a graph. Its nodes are represented by centers of the cells and its links are common sides between adjacent cells.

In case of exact cell decomposition there are two types of cells

- white cells correspond to the collision-free space
- black cells correspond to the collision space

Then given initial and final configurations, search of consecutive transition from one white cell to another one is carrying out to connect these two configurations and avoid all the black cells.

Path Planning: Exact Cell Decomposition Approach



Steps of Exact Cell Decomposition Approach

Path Planning: Approximate Cell Decomposition Approach

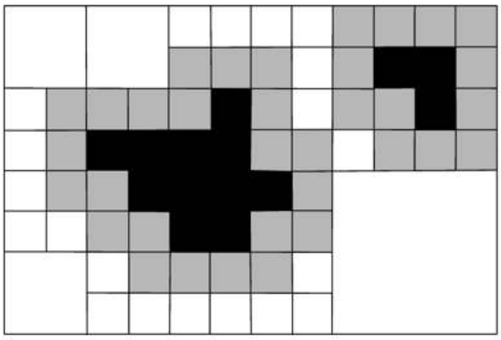
> **Approximate Cell Decomposition**
> Difference of the *approximate cell decomposition* with respect to its "exact" version is that instead of the whole configuration space its subset is divided on cells. So, the remaining space could include also slight parts of collision-free space, which is caused by complex shape of the collision space.

In case of approximate cell decomposition there are two types of cells

- white cells correspond to the collision-free space
- black cells correspond to the collision space
- gray cells correspond to the both spaces

While searching a path could pass both white and gray cells. If it touches grays cells, additional cell decomposition should be carried out until the path connecting initial and final configurations goes through white cells only.

Path Planning: Approximate Cell Decomposition Approach



Approximate Cell Decomposition

Path Planning: Potential Field Approach

Potential Field Approach

he robot is considered as a material point moving in a configuration space under influence of a potential field function $P(q)$. It has attraction component $P_a(q)$ assigned with the final configuration and repulsive component $P_r(q)$ assigned with the collision space

$$P(q) = P_a(q) + P_r(q). \tag{7}$$

Path Planning: Potential Field Approach

Set the global minimum of the function $P(q)$ as the attraction component $P_a(q)$

$$P_a(q) = \frac{1}{2} k_a ||q - q_d||^2, \tag{8}$$

where q, q_d are the current and desired configurations, respectively, k_a is the scaling factor.

The repulsive component $P_r(q)$ ensures singularity of the function $P(q)$ when the material point is approaching the collision space

$$P_r(q) = \begin{cases} \frac{1}{2} k_r \left(\frac{1}{\delta(q)} - \frac{1}{\delta_0} \right)^2 & \text{if} \quad \delta(q) \leq \delta_0, \\ 0 & \text{if} \quad \delta(q) > \delta_0, \end{cases} \tag{9}$$

where k_r is the scaling factor, $\delta(q)$ is the shortest distance from the current configuration to the collision space, δ_0 is the minimum value.

Path Planning: Potential Field Approach

The gradient descent algorithm can be used to plan a path

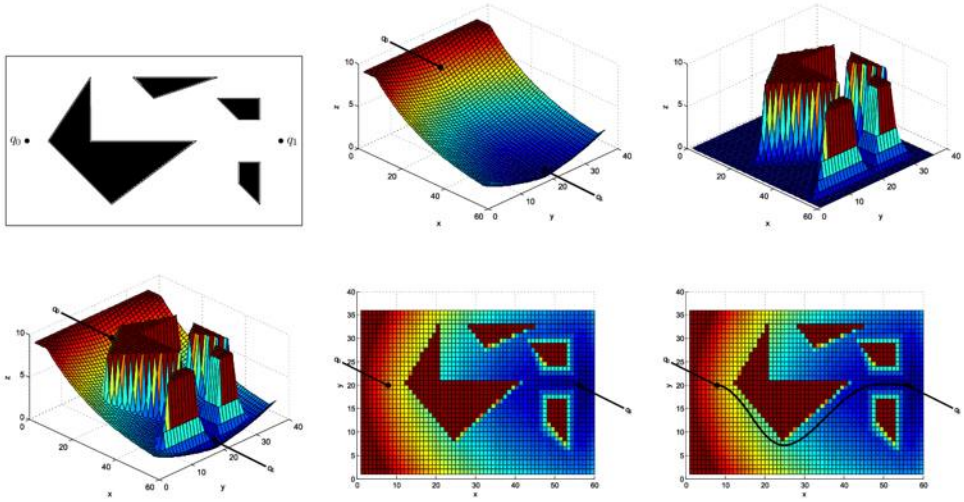
$$q_{j+1} = q_j - \gamma_j \nabla P(q_j), \tag{10}$$

where $\nabla P(q) = \begin{bmatrix} \frac{\partial P}{\partial q_1} & \frac{\partial P}{\partial q_2} \dots & \frac{\partial P}{q_n} \end{bmatrix}^T$, γ_j is a iterative step, which can be either fixed, fractioned, or calculated in the direction of the fastest descent as

$$\gamma_j = \operatorname{argmin}_j P(q_j - \gamma \nabla P(q_j)). \tag{11}$$

The main disadvantage of the potential field approach is possibility to stuack at the local minimum instead of the global one. So called random motion approach is used to avoid this issue.

Path Planning: Potential Field Approach



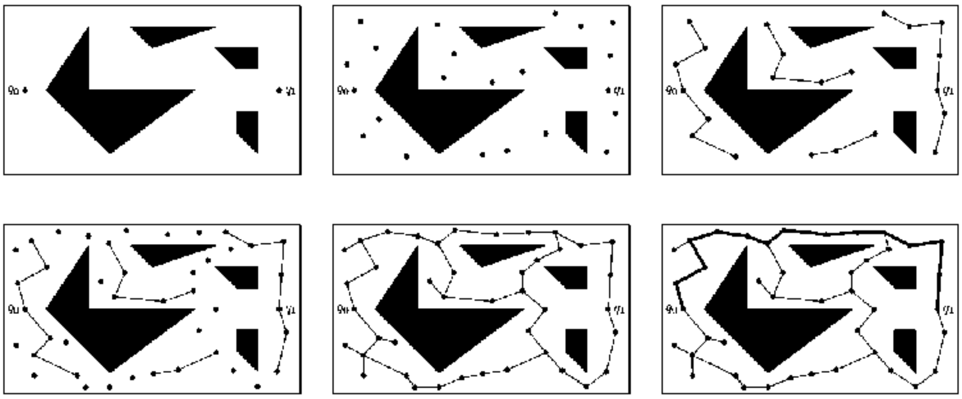
Steps of Potential Field Approach

Path Planning: Probabilistic Roadmap Approach

> **Probabilistic Roadmap Approach**
> Probabilistic roadmap approach is useful for fast path generation. It is based on the usage of random samples from the configuration space.

1. Several nodes (samples) are chosen randomly from the configuration space. Each node is assigned with a particular configuration.
2. Adjacent nodes are being connected between each other within the specified norm in the configuration space.
3. The fisrt two steps are repeated to cover sufficiently large area between the initial and final configurations.
4. A cosequtive set of samples are chosen to connect the initial and final configurations.

Path Planning: Probabilistic Roadmap Approach



Steps of Probabilistic Roadmap Approach

Trajectory Planning: Spline Functions Approach

Spline Functions Approach

The idea of this approach is to interpolate generalized coordinates, velocities and accelerations between the reference points using the polynomials of the form

$$q_i(t) = a_{l,i}t^l + a_{l-1,i}t^{l-1} + \cdots + a_{2,i}t^2 + a_{1,i}t + a_{0,i}, \quad (12)$$

$$\dot{q}_i(t) = la_{l,i}t^{l-1} + (l-1)a_{l-1,i}t^{l-2} + \cdots + 2a_{2,i}t + a_{1,i}, \quad (13)$$

$$\ddot{q}_i(t) = l(l-1)a_{l,i}t^{l-2} + (l-1)(l-2)a_{l-1,i}t^{l-3} + \cdots + 2a_{2,i}, \quad (14)$$

where the degree l and coefficients a_{ji}, $j = \{1, 2, \ldots, l\}$ are calculated depending on the constraints and continuity requirements on the trajectory.

1. Divide the whole trakectory on several elementary subtrajectories.
2. Compute relative time functions τ_i for each subtrjactory.
3. Apply constraints and continuity requirements on the trajectory.
4. Determine the highst polynomial degree for each subtrajectory.
5. Solve matrix equation to compute coefficients of all the polynomials.

Trajectory Planning: Single Subtrajectory Case

Only initial and final configurations are given. No intermediate requirements.

Consider the following constraints for each link of the robot

$$q_i(t_0) = \vartheta_0, \quad \dot{q}_i(t_0) = v_0, \quad \ddot{q}_i(t_0) = \alpha_0, \tag{15}$$

$$q_i(t_1) = \vartheta_1, \quad \dot{q}_i(t_1) = v_1, \quad \ddot{q}_i(t_1) = \alpha_1. \tag{16}$$

Choose the polynomial to interpolate intermediate values of the generalized coordinates

$$\vartheta(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0. \tag{17}$$

Calculate the first and second derivatives of this polynomial to interpolate values of generalized velocities and accelerations

$$\dot{\vartheta}(t) = v(t) = 5a_5 t^4 + 4a_4 t^3 + 3a_3 t^2 + 2a_2 t + a_1, \tag{18}$$

$$\ddot{\vartheta}(t) = \alpha(t) = 20a_5 t^3 + 12a_4 t^2 + 6a_3 t + 2a_2. \tag{19}$$

Trajectory Planning: Single Subtrajectory Case

Write the system of equations taking into account the imposed constraits and continuity requirements as follows

$$\begin{cases} \vartheta_0 = a_5 t_0^5 + a_4 t_0^4 + a_3 t_0^3 + a_2 t_0^2 + a_1 t_0 + a_0, \\ v_0 = 5a_5 t_0^4 + 4a_4 t_0^3 + 3a_3 t_0^2 + 2a_2 t_0 + a_1, \\ \alpha_0 = 20a_5 t_0^3 + 12a_4 t_0^2 + 6a_3 t_0 + 2a_2, \\ \vartheta_1 = a_5 t_1^5 + a_4 t_1^4 + a_3 t_1^3 + a_2 t_1^2 + a_1 t_1 + a_0, \\ v_1 = 5a_5 t_1^4 + 4a_4 t_1^3 + 3a_3 t_1^2 + 2a_2 t_1 + a_1, \\ \alpha_1 = 20a_5 t_1^3 + 12a_4 t_1^2 + 6a_3 t_1 + 2a_2. \end{cases} \tag{20}$$

Rewrite this system in matrix form as

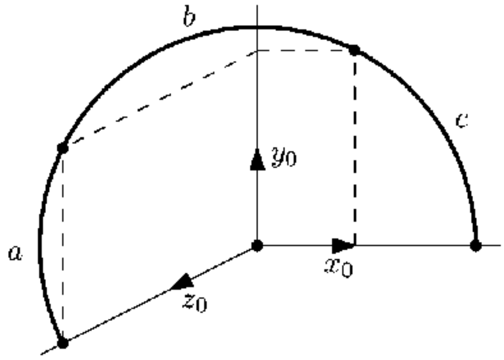
$$\underbrace{\begin{bmatrix} \vartheta_0 \\ v_0 \\ \alpha_0 \\ \vartheta_1 \\ v_1 \\ \alpha_1 \end{bmatrix}}_{\varrho} = \underbrace{\begin{bmatrix} t_0^5 & t_0^4 & t_0^3 & t_0^2 & t_0 & 1 \\ 5t_0^4 & 4t_0^3 & 3t_0^2 & 2t_0 & 1 & 0 \\ 20t_0^3 & 12t_0^2 & 6t_0 & 2 & 0 & 0 \\ t_1^5 & t_1^4 & t_1^3 & t_1^2 & t_1 & 1 \\ 5t_1^4 & 4t_1^3 & 3t_1^2 & 2t_1 & 1 & 0 \\ 20t_1^3 & 12t_1^2 & 6t_1 & 2 & 0 & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}}_{\varsigma}, \tag{21}$$

from which the vector of unknown coefficients can be easily expressed as

Trajectory Planning: Multiple Subtrajectory Case

Consider a trajectory comprised of three subtrajectories as follows

- Leaving (a)
- Transition (b)
- Approach (c)



Trajectory comprised of three segments

Trajectory Planning: Multiple Subtrajectory Case

Consider the following constraints for each link of the robot

$$q_i(t_0) = \vartheta_0, \quad q_i(t_1) = \vartheta_1, \quad q_i(t_2) = \vartheta_2, \quad q_i(t_3) = \vartheta_3, \tag{23}$$

$$\dot{q}_i(t_0) = \upsilon_0, \quad \ddot{q}_i(t_0) = \alpha_0, \quad \dot{q}_i(t_3) = \upsilon_3, \quad \ddot{q}_i(t_3) = \alpha_3. \tag{24}$$

Use the relative time functions for each subtrajectory

$$\tau_a = \frac{t - t_0}{t_1 - t_0}, \qquad \tau_b = \frac{t - t_1}{t_2 - t_1}, \qquad \tau_c = \frac{t - t_2}{t_3 - t_2}, \tag{25}$$

where t_0, t_1, t_2, t_3 are the given time moments of passing all the reference configurations.

Impose continuity requirements to get a smooth trajectory

$$\upsilon_a(1) = \upsilon_b(0), \quad \alpha_a(1) = \alpha_b(0), \quad \upsilon_b(1) = \upsilon_c(0), \quad \alpha_b(1) = \alpha_c(0). \tag{26}$$

Taking into account the relative time functions rewrite constraints on the trajectory

$$\vartheta_a(0) = \vartheta_0, \quad \vartheta_b(0) = \vartheta_1, \quad \vartheta_c(0) = \vartheta_2, \tag{27}$$

$$\vartheta_a(1) = \vartheta_1, \quad \vartheta_b(1) = \vartheta_2, \quad \vartheta_c(1) = \vartheta_3, \tag{28}$$

and continuity requirementsas follows

Trajectory Planning: Multiple Subtrajectory Case

Choose the polynomial to interpolate intermediate values of the generalized coordinates within each subtrajectory

$$\vartheta_a(\tau_a) = a_4\tau_a^4 + a_3\tau_a^3 + a_2\tau_a^2 + a_1\tau_a + a_0, \quad (30)$$

$$\vartheta_b(\tau_b) = b_3\tau_b^3 + b_2\tau_b^2 + b_1\tau_b + b_0, \quad (31)$$

$$\vartheta_c(\tau_c) = c_4\tau_c^4 + c_3\tau_c^3 + c_2\tau_c^2 + c_1\tau_c + c_0, \quad (32)$$

Calculate the first derivative of these polynomials to interpolate values of generalized velocities

$$\dot\vartheta_a(\tau_a) = v_a(\tau_a) = 4a_4\tau_a^3 + 3a_3\tau_a^2 + 2a_2\tau_a + a_1, \quad (33)$$

$$\dot\vartheta_b(\tau_b) = v_b(\tau_b) = 3b_3\tau_b^2 + 2b_2\tau_b + b_1, \quad (34)$$

$$\dot\vartheta_c(\tau_c) = v_c(\tau_c) = 4c_4\tau_c^3 + 3c_3\tau_c^2 + 2c_2\tau_c + c_1, \quad (35)$$

Calculate the second derivative of these polynomials to interpolate values of generalized accelerations

$$\ddot\vartheta_a(\tau_a) = \alpha_a(\tau_a) = 12a_4\tau_a^2 + 6a_3\tau_a + 2a_2, \quad (36)$$

$$\ddot\vartheta_b(\tau_b) = \alpha_b(\tau_b) = 6b_3\tau_b + 2b_2, \quad (37)$$

$$\ddot\vartheta_c(\tau_c) = \alpha_c(\tau_c) = 12c_4\tau_c^2 + 6c_3\tau_c + 2c_2, \quad (38)$$

Trajectory Planning: Multiple Subtrajectory Case

Write the system of equations taking into account the imposed constraits and continuity requirements as follows

$$\begin{cases} \vartheta_0 = a_0, \\ v_0 = a_1, \\ \alpha_0 = 2a_2, \\ \vartheta_1 = a_4 + a_3 + a_2 + a_1 + a_0, \\ \vartheta_1 = b_0, \\ 0 = 4a_4 + 3a_3 + 2a_2 + a_1 - b_1, \\ 0 = 12a_4 + 6a_3 + 2a_2 - 2b_2, \\ \vartheta_2 = b_3 + b_2 + b_1 + b_0, \\ \vartheta_2 = c_0, \\ 0 = 3b_3 + 2b_2 + b_1 - c_1, \\ 0 = 6b_3 + 2b_2 - 2c_2, \\ \vartheta_3 = c_4 + c_3 + c_2 + c_1 + c_0, \\ v_3 = 4c_4 + 3c_3 + 2c_2 + c_1, \\ \alpha_3 = 12c_4 + 6c_3 + 2c_2. \end{cases} \quad (39)$$

Trajectory Planning: Multiple Subtrajectory Case

Rewrite this system in matrix form as

$$\underbrace{\begin{bmatrix} \vartheta_0 \\ v_0 \\ \alpha_0 \\ \vartheta_1 \\ \vartheta_1 \\ 0 \\ 0 \\ \vartheta_2 \\ \vartheta_2 \\ 0 \\ 0 \\ \vartheta_3 \\ v_3 \\ \alpha_3 \end{bmatrix}}_{\varrho} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 6 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 6 & 2 & 0 & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_0 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}}_{\varsigma}, \qquad (40)$$

from which the vector of unknown coefficients can be easily expressed as

$$\varsigma = T^{-1}\varrho. \qquad (41)$$

Arc Approximation Algorithm of Spatial Movements

Research Objective

This study focuses on spatial motion planning algorithms, which allows to characterize sophisticated reference paths in 3D space and simplify the way how they can be given. The key point used in this study is approximation of a sequence of points by a sequence of arcs within a specified δ-region.

In industry such algorithm can be applied for such tasks as surface finishing, engraving and welding. The last operation represents the main interest of this research.

Problem

Mitsubishi RV-3SDB

Objective
The purpose is automated code generating to move the end-effector along some counters specified by the input bitmap image or 3D model.

After extracting coordinates of initial points sequence they already can be programmed using trivial point-to-point motion, but it might lead to some issues.

- significant input data (robot controller overload)
- decrease of the motion velocity (reconfiguration at each reference point)

Arc Approximation Algorithm

Basic Idea
This approach is based on the feasibility of the standard software to move the end-effector along an arc, specified with only three points. This basic motion provided by the internal software is more natural then complex combinations of multiple linear point-to-point movements. As a result, the robot reconfigures only three times at the reference points forming this arc. Such solution allows to reduce the code size and increase the velocity.

Planar Planning

Consider three reference points

$$p_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad p_3 = \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}. \tag{42}$$

All intermediate points between p_1, p_2 and p_3 should belong to a corresponding arc within some δ_{arc}-region.

Consider two lines p_1–p_2 and p_2–p_3. In order to find coordinates of the arc center $c = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$ consider three cases.

Case 1

If $x_2 = x_3$ and $x_1 \neq x_2$ then

$$y_c = \frac{y_2 + y_3}{2}, \quad x_c = -k_1 \frac{y_c - (y_1 + y_2)}{2} + \frac{x_1 + x_2}{2},$$

where k_1 is the slope of the line $(x_1; y_1)$–$(x_2; y_2)$ given by $k_1 = \frac{y_2 - y_1}{x_2 - x_1}$.

Case 2

If $x_1 = x_2$ and $x_2 \neq x_3$ then

$$y_c = \frac{y_1 + y_2}{2}, \quad x_c = -k_2 \frac{y_c - (y_2 + y_3)}{2} + \frac{x_2 + x_3}{2},$$

where k_2 is the slope of the line $(x_2; y_2)$–$(x_3; y_3)$ given by $k_2 = \frac{y_3 - y_2}{x_3 - x_2}$.

Case 3

If all x-coordinates are distinct, then

$$x_c = \frac{k_1 k_2 (y_1 - y_3) + k_2 (x_1 + x_2) - k_1 (x_2 + x_3)}{2(k_2 - k_1)}, \tag{43}$$

$$y_c = -\frac{x_c - \frac{x_1 + x_2}{2}}{k_1} + \frac{y_1 + y_2}{2}, \tag{44}$$

where k_1 and k_2 are given above.

Calculate distance from a forth point $p_4 = \begin{bmatrix} x_4 \\ y_4 \end{bmatrix}$ to the arc formed by p_1, p_2 and p_3 as follows

$$d_{arc} = \left| \sqrt{(x_c - x_4)^2 + (y_c - y_4)^2} - r \right|, \tag{45}$$

where $r = \sqrt{(x_1 - x_c)^2 + (y_1 - y_c)^2}$ is the radius of the arc.

As a result we get a sequence of arcs each specified by three consecutive points. Such point list can be used together with the operator MVR P1 P2 P3, which allows to move along an arc specified by three reference points.

Spatial Planning

Consider three points that do not lie on the same line. Coordinates of vectors specified in the Cartesian space are defined as

$$p_1^0 = \begin{bmatrix} x_1^0 \\ y_1^0 \\ z_1^0 \end{bmatrix}, \quad p_2^0 = \begin{bmatrix} x_2^0 \\ y_2^0 \\ z_2^0 \end{bmatrix}, \quad p_3^0 = \begin{bmatrix} x_3^0 \\ y_3^0 \\ z_3^0 \end{bmatrix}. \tag{46}$$

Consider two coordinate systems denoted as $x_0 y_0 z_0 o_0$ and $x_1 y_1 z_1 o_1$. Derive a normal to the plane $x_1 y_1 o_1$ through a cross product

$$n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \left(p_2^0 - p_1^0\right) \times \left(p_3^0 - p_1^0\right). \tag{47}$$

Then calculate a unit vector

$$z = \begin{bmatrix} z_x \\ z_y \\ z_z \end{bmatrix} = \frac{n}{\sqrt{n_x^2 + n_y^2 + n_z^2}}. \tag{48}$$

Compute the rotational transformation as $R_1^0 = R_{z,\alpha} R_{y,\beta}$, where the angles α and β can be calculated as follows

$$\alpha = \mathrm{atan2}\left(\frac{z_y}{\sqrt{z_x^2 + z_y^2}}, \frac{z_x}{\sqrt{z_x^2 + z_y^2}} \right), \quad \beta = \mathrm{atan2}\left(\sqrt{z_x^2 + z_y^2}, z_z \right).$$

Substitute α and β into the rotation matrices around $z-$ and y-axes

$$R_1^0 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}. \tag{49}$$

Calculate coordinates of the reference points with respect to the local coordinate system using the rotation matrix

$$p_1^1 = R_0^1 p_1^0, \quad p_2^1 = R_0^1 p_2^0, \quad p_3^1 = R_0^1 p_3^0, \tag{50}$$

where $R_0^1 = [R_1^0]^T$.

Denote coordinates as follows

$$p_1^1 = \begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \end{bmatrix}, \quad p_2^1 = \begin{bmatrix} x_2^1 \\ y_2^1 \\ z_2^1 \end{bmatrix}, \quad p_3^1 = \begin{bmatrix} x_3^1 \\ y_3^1 \\ z_3^1 \end{bmatrix}. \tag{51}$$

In order to find coordinates of the arc center $c^1 = \begin{bmatrix} x_c^1 \\ y_c^1 \\ z_c^1 \end{bmatrix}$ consider three cases.

Case 1

If $x_2^1 = x_3^1$ and $x_1^1 \neq x_2^1$ then

$$y_c^1 = \frac{y_2^1 + y_3^1}{2}, \quad x_c^1 = -k_1 \frac{y_c^1 - (y_1^1 + y_2^1)}{2} + \frac{x_1^1 + x_2^1}{2},$$

where k_1 is the slope of the line $(x_1^1; y_1^1)$–$(x_2^1; y_2^1)$ given by $k_1 = \frac{y_2^1 - y_1^1}{x_2^1 - x_1^1}$.

Case 2

If $x_1^1 = x_2^1$ and $x_2^1 \neq x_3^1$ then

$$y_c^1 = \frac{y_1^1 + y_2^1}{2}, \quad x_c^1 = -k_2 \frac{y_c^1 - (y_2^1 + y_3^1)}{2} + \frac{x_2^1 + x_3^1}{2},$$

where k_2 is the slope of the line $(x_2^1; y_2^1)$–$(x_3^1; y_3^1)$ given by $k_2 = \frac{y_3^1 - y_2^1}{x_3^1 - x_2^1}$.

Case 3

If all x-coordinates are distinct, then

$$x_c^1 = \frac{k_1 k_2 (y_1^1 - y_3^1) + k_2 (x_1^1 + x_2^1) - k_1 (x_2^1 + x_3^1)}{2(k_2 - k_1)}, \tag{52}$$

$$y_c^1 = -\frac{x_c^1 - \frac{x_1^1 + x_2^1}{2}}{k_1} + \frac{y_1^1 + y_2^1}{2}, \tag{53}$$

where k_1 and k_2 are given above.

The third z-coordinate can be derived trivially as

$$z_c^1 = z_1^1 = z_2^1 = z_3^1. \tag{54}$$

Express coordinates of the center with respect to the base coordinate system

$$c^0 = \begin{bmatrix} x_c^0 \\ y_c^0 \\ z_c^0 \end{bmatrix} = R_1^0 c^1. \tag{55}$$

The equation of a plane is given as

$$n_x x + n_y y + n_z z + n_0 = 0, \tag{56}$$

where $n_0 = -(n_x x_3^0 + n_y y_3^0 + n_z z_3^0)$.

Distances from a forth point $p_4 = \begin{bmatrix} x_4^0 \\ y_4^0 \\ z_4^0 \end{bmatrix}$ respectively to the plane d_{plane} and to the arc formed by p_1, p_2 and p_3 can be computed as

$$d_{plane} = \frac{|n_x x_4^0 + n_y y_4^0 + n_z z_4^0 + n_0|}{\sqrt{n_x^2 + n_y^2 + n_z^2}}, \tag{57}$$

$$d_{arc} = \left| \sqrt{(x_c^0 - x_4^0)^2 + (y_c^0 - y_4^0)^2 + (z_c^0 - z_4^0)^2} - r \right|, \tag{58}$$

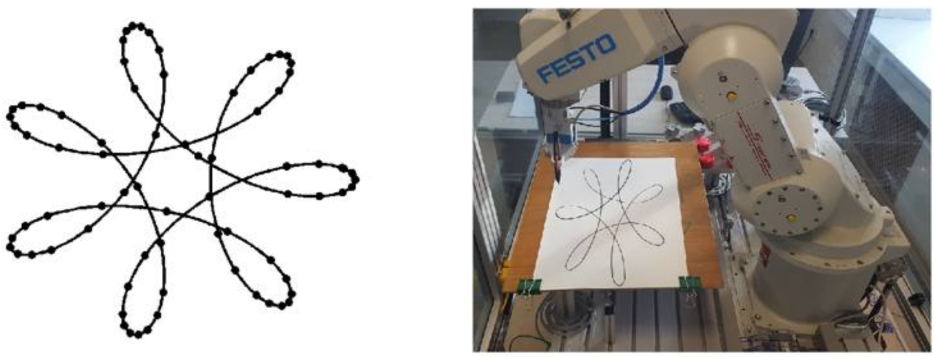
where $r = \sqrt{(x_1^0 - x_c^0)^2 + (y_1^0 - y_c^0)^2 + (z_1^0 - z_c^0)^2}$ is the radius of the arc.

Then all points should be processed and checked on belonging them to a particular plane and arc within the specified δ_{plane}- and δ_{arc}-regions, respectively. As a result of this procedure, a sequence of three-points-sets each specifying a particular arc should be obtained.

Experimental Approval

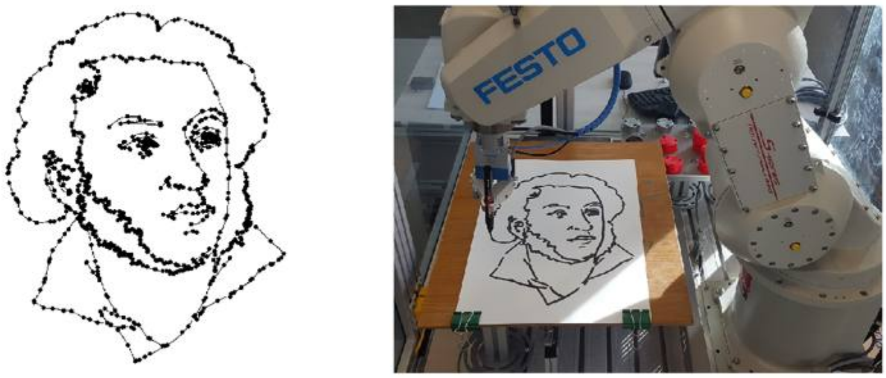
Experimental Approval

Experimental Results: Planar Planning



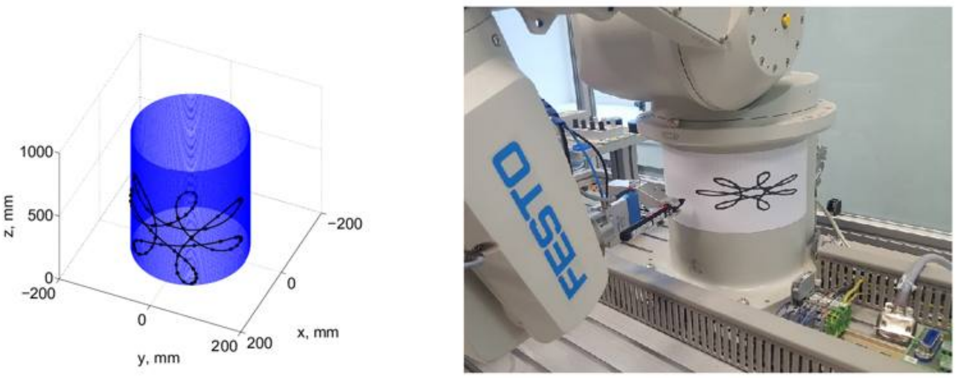
A hypotrochoid drawn by the robot on a flat surface

Experimental Results: Planar Planning



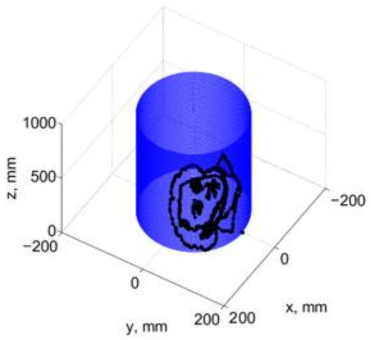
A portrait of Alexander Pushkin drawn by the robot on a flat surface

Experimental Results: Spatial Planning



A hypotrochoid drawn by the robot on the a curved (cylindrical) surface

Experimental Results: Spatial Planning



A portrait of Alexander Pushkin drawn by the robot on a curved (cylindrical) surface

Summary

- Reference motion can be programmed manually using a teach pendant or automatically using some path planning algorithm
- Once a path is generated, its intermediate positions, velocities and accelerations should be interpolated
- Advanced algorithms for spatial movement planning can be designed for industrial applications
- The next step is control design to make the robot to track the reference trajectory

Control design for industrial robots

Control Design for Industrial Robots

Dr. Oleg Borisov

PD Controller

Consider the control plant specified by the transfer function. We introduce a proportional-differential (PD) controller with a transfer function

$$R(s) = k_p + k_d s. \tag{1}$$

We calculate the transfer function of a closed-loop system

$$\begin{aligned} W(s) &= \frac{R(s)P(s)}{1 + R(s)P(s)} = \frac{\frac{k_p - k_d s}{Js^2 + Ks}}{1 + \frac{k_p + k_d s}{Js^2 + Ks}} = \\ &= \frac{k_p + k_d s}{Js^2 + (K + k_d)s + k_p}. \end{aligned} \tag{2}$$

PD Controller scheme

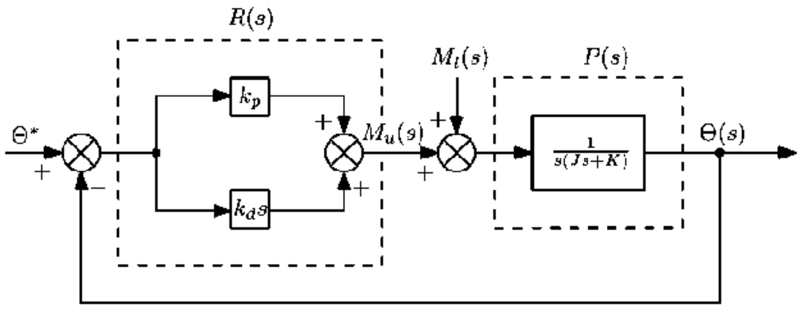


Figure 1: Simulation scheme of a closed-loop system with PD controller

Further, with known parameters of the object J & K, based on the roots of the characteristic polynomial of the transfer function $Js^2 + (K + k_d)s + k_p$, it is possible to calculate such coefficients of the PD controller k_p & k_d to ensure the required quality indicators of the closed system.

PID Controller

Consider the control plant given by the transfer function. We introduce a proportional-integral-differential (PID) controller with a transfer function

$$R(s) = k_p + k_i \frac{1}{s} + k_d s. \quad (3)$$

With structural transformations we express the output variable

$$\Theta(s) = \frac{R(s)P(s)}{1 + R(s)P(s)}\Theta^*(s) + \frac{P(s)}{1 + R(s)P(s)}M_l(s). \quad (4)$$

We calculate the transfer function of a closed-loop system

$$W(s) = \frac{R(s)P(s)}{1 + R(s)P(s)} = \frac{\frac{k_d s^2 + k_p s + k_i}{Js^3 + Ks^2}}{1 + \frac{k_d s^2 + k_p s + k_i}{Js^3 + Ks^2}} =$$

$$= \frac{k_d s^2 + k_p s + k_i}{Js^3 + (K + k_d)s^2 + k_p s + k_i}. \quad (5)$$

PID Controller scheme

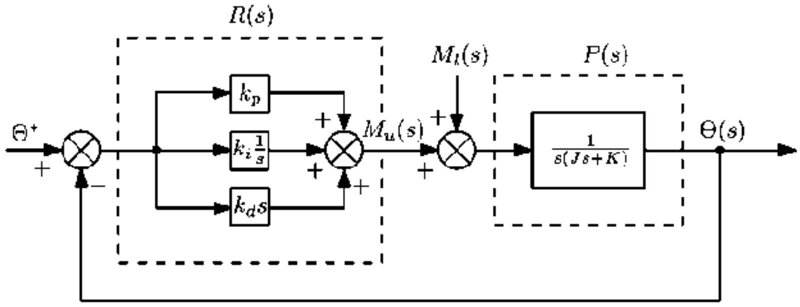


Figure 2: Simulation scheme of a closed-loop system with a PID controller

Further, with known parameters of the object J & K, based on the roots of the characteristic polynomial of the transfer function $Js^3 + (K + k_d)s^2 + k_p s + k_i$, it is possible to calculate such coefficients of the PID regulator k_p, k_i & k_d in order to ensure the required quality indicators of the closed system.

Robust control

Let us write down the consequtive compensator in the form of a transfer function

$$R(s) = k\gamma_0\sigma^{\rho-1}\frac{\alpha(s)}{\gamma(s)}, \tag{6}$$

where ρ — relative degree of the plant, k & $\sigma > k$ — tuning parameters of the controller, $\alpha(s)$ an arbitrary Hurwitz polynomial of degree $\rho - 1$, $\gamma(s)$ — Hurwitz polynomial of the form

$$\gamma(s) = s^{\rho-1} + \sigma\gamma_{\rho-2}s^{\rho-2} + \cdots + \sigma^{\rho-2}\gamma_1 s + \sigma^{\rho-1}\gamma_0. \tag{7}$$

Robust control in closed-loop system

Consider the control object specified by the transfer function. Its relative degree is $\rho = 2$, so, chosen $\alpha(s) = s + 1$ & $\gamma_0 = 1$, rewrite the regulator (6) like

$$R(s) = \frac{k\sigma s + k\sigma}{s + \sigma}. \tag{8}$$

Transfer function of a closed-loop system is

$$\begin{aligned} W(s) &= \frac{R(s)P(s)}{1 + R(s)P(s)} = \frac{\frac{k\sigma s + k\sigma}{(s+\sigma)(Js^2+Ks)}}{1 + \frac{k\sigma s + k\sigma}{(s+\sigma)(Js^2+Ks)}} = \\ &= \frac{k\sigma s + k\sigma}{(s+\sigma)(Js^2 + Ks) + k\sigma s + k\sigma}. \end{aligned} \tag{9}$$

Robust control scheme

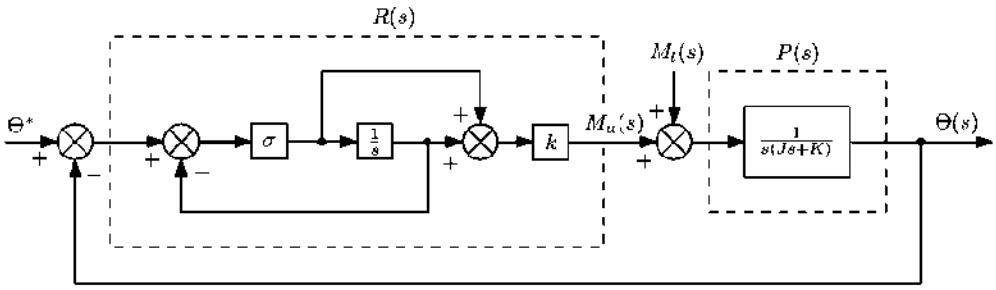


Figure 3: Simulation scheme of a closed-loop system with a consecutive compensator

The characteristic polynomial of the transfer function (9) contains unknown parameters of the plat, but due to the robustness of the regulator (8), for sufficiently large coefficients k & σ exponential stability is attained.

Robust control extended

Adding the integral component we rewrite the regulator (6)

$$R(s) = k\gamma_0\sigma^{\rho-1}\frac{\beta(s)}{s\gamma(s)}, \tag{10}$$

where $\beta(s)$ — Hurwitz polynomial of degree ρ.

Having chosen $\beta(s) = s^2 + s + 1$ & $\gamma_0 = 1$ rewrite the regulator (10) like

$$R(s) = \frac{k\sigma s^2 + k\sigma s + k\sigma}{s^2 + \sigma s}. \tag{11}$$

Transfer function of a closed-loop system

$$\begin{aligned} W(s) &= \frac{R(s)P(s)}{1 + R(s)P(s)} = \frac{\frac{k\sigma s^2 + k\sigma s + k\sigma}{(s^2+\sigma s)(Js^2+Ks)}}{1 + \frac{k\sigma s^2 + k\sigma s + k\sigma}{(s^2+\sigma s)(Js^2+Ks)}} = \\ &= \frac{k\sigma s^2 + k\sigma s + k\sigma}{(s^2 + \sigma s)(Js^2 + Ks) + k\sigma s^2 + k\sigma s + k\sigma} \end{aligned} \tag{12}$$

Robust control extended scheme

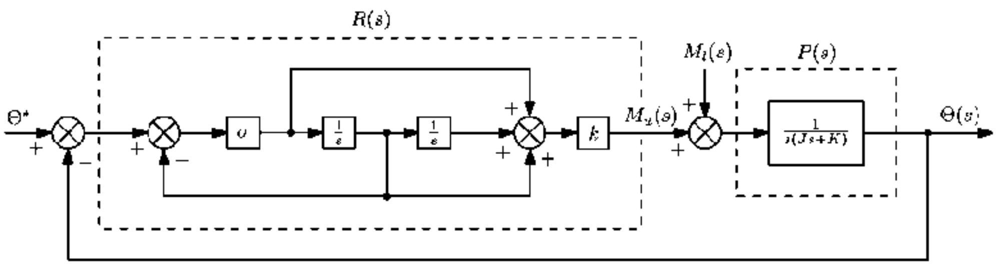


Figure 4: Simulation scheme for a closed-loop system with a consecutive compensator with integral loop

The increased order of astaticism of a system with a transfer function (12) makes it possible to compensate the effect of gravitational forces.

Anti-Windup Control

Saturated input

$$\hat{u}(t) = \operatorname{sat} u(t) = \begin{cases} u_t, & \text{if} \quad u(t) \geq u_t, \\ u(t), & \text{if} \quad u_b < u(t) < u_t, \\ u_b, & \text{if} \quad u(t) \leq u_b, \end{cases} \tag{13}$$

where u_t & u_b — upper and lower limits of the input signal.

Let us write down the control law of the PID controller (3) like

$$u(t) = k_p \tilde{q}(t) + k_i \frac{\tilde{q}(t)}{p} + k_d p \tilde{q}(t), \tag{14}$$

where $p = \frac{d}{dt}$ — differentiation operator, $\tilde{q}(t) = q^* - q(t)$ — error.

Following the antivindap correction method we add to (14) an additional contour

$$u(t) = k_p \tilde{q}(t) + k_i \frac{\tilde{q}(t) + k_u \tilde{u}(t)}{p} + k_d p \tilde{q}(t), \tag{15}$$

where $k_u > 0$ — gain, $\tilde{u}(t) = \hat{u}(t) - u(t)$ — difference signal between saturated and source control.

Anti-Windup Control scheme

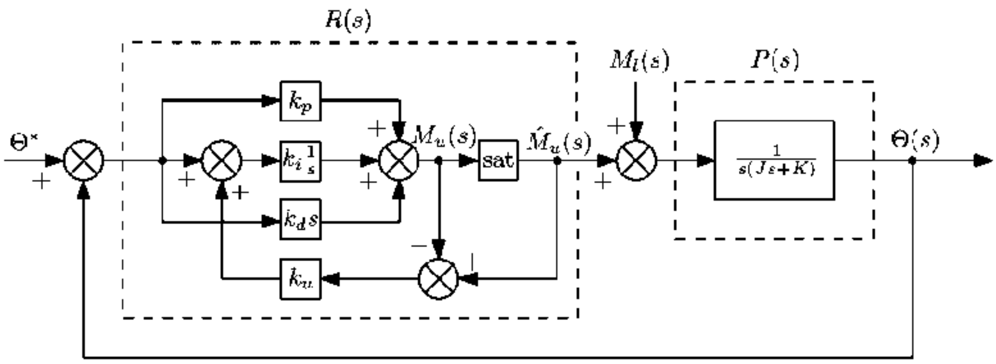


Figure 5: Simulation scheme of a closed-loop system with a PID controller and anti-windup correction

The control law (15) helps to avoid the effect of integral saturation in conditions of limited input.

Anti-Windup Robust Control

Let us write down the control law of a consequtive compensator with an integrated circuit (10) like

$$u(t) = k\frac{\beta(p)}{p}\hat{\tilde{q}}(t), \quad (16)$$

$$\dot{\xi}(t) = \sigma(\Gamma\xi(t) + d\gamma_0\tilde{q}(t)), \quad (17)$$

$$\hat{\tilde{q}}(t) = h^T\xi(t), \quad (18)$$

where $\hat{\tilde{q}}(t)$ — error signal estimation $\tilde{q}(t)$, matrices and vectors Γ, d, h in form

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \dots & -\gamma_{p-1} \end{bmatrix},\ d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},\ h = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (19)$$

Anti-Windup Robust Control

Transform the control law (16), with integrator

$$u(t) = k\frac{\beta(p)}{p}\hat{\tilde{q}}(t) = k\left(\bar{\beta}(p) + \frac{\beta_0}{p}\right)\hat{\tilde{q}}(t) = k\bar{\beta}(p)\hat{\tilde{q}}(t) + k\frac{\beta_0}{p}\hat{\tilde{q}}(t), \quad (20)$$

where $\bar{\beta}(p) = \frac{\beta(p)-\beta_0}{p}$.

Following the anti-windup correction method we add to the (20) an additional contour

$$u(t) = k\bar{\beta}(p)\hat{\tilde{q}}(t) + k\frac{\beta_0}{p}\left(\hat{\tilde{q}}(t) + k_u\tilde{u}(t)\right), \quad (21)$$

where $k_u > 0$ — gain, $\tilde{u}(t) = \hat{u}(t) - u(t)$ — difference signal between saturated and source control.

Having chosen $\beta(p) = p^2 + p + 1$ & $\gamma_0 = 1$, rewrite the regulator (21) like

$$u(t) = kp\hat{\tilde{q}}(t) + k\hat{\tilde{q}}(t) + k\frac{1}{p}(\hat{\tilde{q}}(t) + k_u\tilde{u}(t)). \quad (22)$$

Anti-Windup Robust Control

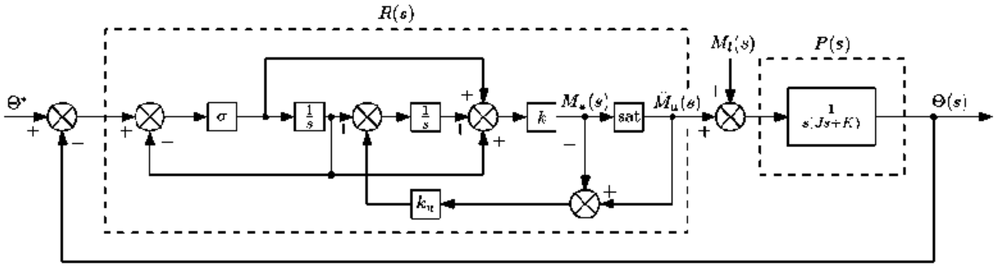


Figure 6: Simulation scheme of a closed-loop system with a consecutive compensator and anti-windup correction

The regulator (22) allows to solve the stabilization problem with the increased order of astaticism in comparison with the regulator (6) and with compensation of the integral saturation effect by means of anti-windup.

Tracking control

Let's express the output signal $\Theta(s)$:

$$\Theta(s) = \frac{R(s)P(s) + F(s)P(s)}{1 + R(s)P(s)}\Theta^*(s) + \frac{P(s)}{1 + R(s)P(s)}M_l(s). \quad (23)$$

We choose the transfer function of direct coupling in the form:

$$F(s) = \frac{1}{P(s)}, \quad (24)$$

then the expression (23) takes the form:

$$\Theta(s) = \Theta^*(s) + +\frac{P(s)}{1 + R(s)P(s)}M_l(s). \quad (25)$$

Tracking control scheme

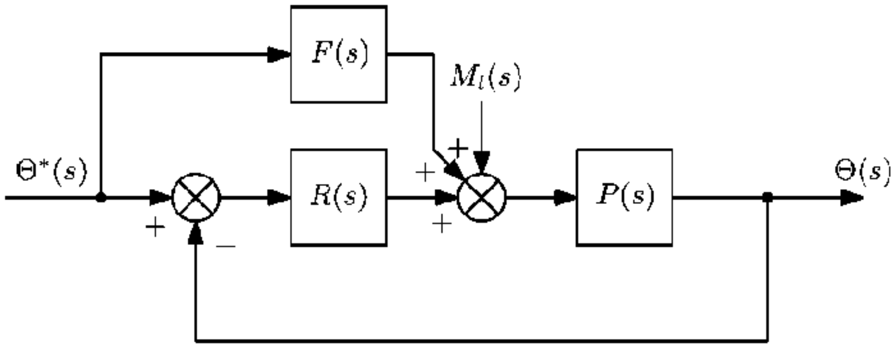


Figure 7: Simulation scheme for closed-loop tracking system

Direct link allows the system to monitor any given trajectory, provided that the system is completely stable. The steady-state error in this case will be due only to the influence of an external perturbation $M_l(s)$.

Multivariable control

Consider the dynamic model of a robotic system

$$\Gamma(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u. \tag{26}$$

Stabilization of desired q^* will be performed with PD controller.

First, for simplicity, we neglect the effect of gravity, assuming that $G(q) = 0$. In view of this model (26) looks like

$$\Gamma(q)\ddot{q} + C(q,\dot{q})\dot{q} = u. \tag{27}$$

We choose the vector of control actions u like

$$u = K_p(q^* - q(t)) - K_d\dot{q}(t) = K_p\tilde{q}(t) + K_d\dot{\tilde{q}}(t), \tag{28}$$

where $\tilde{q}(t) = q^* - q(t)$ — error between the specified configuration and the current one, K_p & K_d looks like

$$K_p = \begin{bmatrix} k_{p,1} & 0 & \dots & 0 \\ 0 & k_{p,2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & k_{p,n} \end{bmatrix}, \quad K_d = \begin{bmatrix} k_{d,1} & 0 & \dots & 0 \\ 0 & k_{d,2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & k_{d,n} \end{bmatrix}. \tag{29}$$

Multivariable control

Substituting the control law (28) to the plant (27) we obtain a model of a closed system

$$\Gamma(q)\ddot{q} + C(q,\dot{q})\dot{q} = K_p\tilde{q}(t) + K_d\dot{\tilde{q}}(t). \tag{30}$$

To analyze the stability of a closed-loop system (30) we consider the candidate Lyapunov function in quadratic form

$$V(t) = \frac{1}{2}\tilde{q}^T K_p \tilde{q} + \frac{1}{2}\dot{\tilde{q}}^T \Gamma \dot{\tilde{q}}. \tag{31}$$

Taking the time derivative of (31) we get

$$\dot{V}(t) = -\dot{\tilde{q}}^T K_d \dot{\tilde{q}} \leq 0, \tag{32}$$

this together with the Lassalle theorem shows the asymptotic stability of a closed system (30).

Multivariable control

When $\dot{V} = 0$ from (32) we can conclude that the generalized velocities and accelerations are zero $\dot{q}(t) = 0$ & $\ddot{q}(t) = 0$. Taking this into account we rewrite the equation of a closed system for $t \to \infty$

$$0 = K_p\tilde{q}(t), \tag{33}$$

from which it follows that $\tilde{q}(t) = q^* - q(t) = 0$ with $t \to \infty$.

The influence of gravity $G(q) \neq 0$ leads to the appearance of a steady error. The PD controller in this case does not provide asymptotic stability. The equation (33) looks like

$$G(q) = K_p\tilde{q}(t). \tag{34}$$

To eliminate the established error we supplement the law of control

$$u = K_p\tilde{q}(t) + K_d\dot{\tilde{q}}(t) + G(q), \tag{35}$$

which makes it possible to provide asymptotic stability with the influence of gravity.

Dynamic of robotic systems

Dynamics of Robotic Systems
Euler-Lagrange Method and Special Cases

Sergey Kolyubin

Outline

- Motivation
- Energy-based Approach - Euler-Lagrange Method
  - Energy calculation
  - Motion equation
- Special Cases
  - Drive dynamics
  - Flexible joints modeling
- Motion Equation in Operational Space

Why Do We Need to Know Dynamics?

- simulation
- defining dynamic constraints
- mechanical design optimization
- trajectory planners and controllers synthesis

Tasks

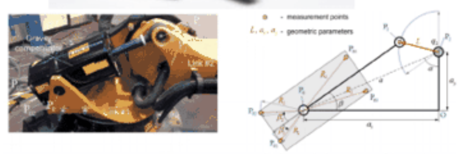
- Forward Dynamics: given desired trajectory (coordinates, velocities, acceleration) find generalized forces/torques
- Inverse Dynamics: given generalized forces/torques, find generated motion (trajectory)

Tasks

- Forward Dynamics: given desired trajectory (coordinates, velocities, acceleration) find generalized forces/torques
- Inverse Dynamics: given generalized forces/torques, find generated motion (trajectory)

Practical tasks

- f/t calculation - find external (control design) and internal (find reaction forces in kinematic pairs)
- performance indicators - find possible cycle time given dynamic constraints
- (serial) manipulators balancing - unload drives in statics
- (parallel) manipulators dynamic balancing - minimize distortions during the motion by placing counter-weights



(a) gravity compensator of KUKA KR-270 TM robot

(b) Geometrical parameters of the gravity compensator and measurement point locations

Tasks

- Forward Dynamics: given desired trajectory (coordinates, velocities, acceleration) find generalized forces/torques
- Inverse Dynamics: given generalized forces/torques, find generated motion (trajectory)

Theoretical sub-tasks

- trajectories calculation
- motion stability analysis
- calculating time response
- identifying critical motion modes

Methods Comparison

- E-L - kinetic and potential energy
  - multibody dynamics as a whole
  - exclude reaction forces between links
  - symbolic form
  - better for analysis
- N-E - forces/torques balance
  - separate equation for each body
  - explicit relations for reaction forces
  - numeric recursion form
  - better for synthesis and real-time applications

By excluding reaction forces and substituting these relations we can derive E-L equations from N-E equations

E-L General Framework

1. select generalized coordinates $q_1, q_2, \dots q_n$
2. derive relations for kinetic $\mathcal{K}$ and potential $\mathcal{P}$ energy as functions of generalized coordinates and its derivatives
3. calculate system Lagrangian $\mathcal{L}$
4. derive motion equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \tau_k, k = 1, 2, \dots n \tag{1}$$

τ_k is a generalized force/torque

Full Kinetic Energy

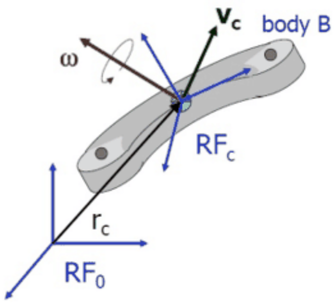


Figure 1: ©DeLuca

Konig theorem
Full energy consist of an energy assoc. with body CoM motion and relative body motion around it CoM

$$\mathcal{K} = \frac{1}{2}m|v|^2 + \frac{1}{2}\omega^T\mathcal{I}\omega$$

where m is a body mass, v and ω are linear and rotational velocities vectors, $\mathcal{I}$ is an inertia tensor

All values in the same CF

Formula for rotational velocity

$$\omega \leftarrow S(\omega) = \dot{R}(t)R^T(t),$$

where R is a rotation matrix from body frame to inertial frame

Kinetic Energy of n-links Robot

Sum of kinetic energy of linear and rotational motions

$$\mathcal{K} = \tfrac{1}{2}m\,|v_c|^2 + \tfrac{1}{2}\omega^T\mathcal{I}\omega$$

CoM velocities

- $v_c = \dot{r}_c$ and ω are functions of generalized coordinates q and velocities $\dot{q}$

Kinetic Energy of n-links Robot

Sum of kinetic energy of linear and rotational motions

$$\mathcal{K} = \tfrac{1}{2} m \left|v_c\right|^2 + \tfrac{1}{2} \omega^T \mathcal{I} \omega$$

CoM velocities

- $v_c = \dot{r}_c$ and ω are functions of generalized coordinates q and velocities $\dot{q}$

Relations can be computed via Jacobian assoc. with links CoMs

$$v_{c,i} = J_{v_i}(q)\dot{q}, \quad \omega_i = J_{\omega_i}(q)\dot{q}$$

Kinetic Energy of n-links Robot

Sum of kinetic energy of linear and rotational motions

$$\mathcal{K} = \tfrac{1}{2} m \left|v_c\right|^2 + \tfrac{1}{2} \omega^T \mathcal{I} \omega$$

CoM velocities

- $v_c = \dot{r}_c$ and ω are functions of generalized coordinates q and velocities $\dot{q}$

Relations can be computed via Jacobian assoc. with links CoMs

$$v_{c,i} = J_{v_i}(q)\dot{q}, \quad \omega_i = J_{\omega_i}(q)\dot{q}$$

Robot kinetic energy

$$\mathcal{K} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

Recurrent Velocities Formulas

- rotation (angular) velocities

$$\omega_i = \left(R_i^{i-1}(q_i)\right)^T [\omega_{i-1} + (1-\sigma_i)\dot{q}_i z_{i-1}] = \left(R_i^{i-1}(q_i)\right)^T \omega_i^{i-1}, \quad (2)$$

where $R_i^{i-1}(q_i)$ is a rotation matrix for neighbor CFs $O_{i-1}x_{i-1}y_{i-1}z_{i-1}$ and $O_i x_i y_i z_i$,

$$\sigma_i = \begin{cases} 0, \textit{for rotational joint}, \\ 1, \textit{for prismatic joint}, \end{cases}$$

$z_{i-1} = [0\,0\,1]^T$ is a vector of z axis coord. if D-H convention used, ω_i^{i-1} is a rotational velocity of i-th link with respect to CF $O_{i-1}x_{i-1}y_{i-1}z_{i-1}$

- linear velocities

$$v_{c,i} = v_i + \omega_i \times r_{c,i}, \quad (3)$$

where

$$v_i = \left(R_i^{i-1}(q_i)\right)^T \left[v_{i-1} + \sigma_i \dot{q}_i z_{i-1} + \omega_i^{i-1} \times r_{i-1,i}^{i-1}\right] \quad (4)$$

denotes linear velocity of a CF origin O_i, $r_{c,i}$ is a CoM vector for i-th link with respect to O_i, $r_{i-1,i}^{i-1}$ are coordinates of radius-vectors from O_{i-1} to O_i with respect to CF $O_{i-1}x_{i-1}y_{i-1}z_{i-1}$.

Potential Energy of n-links Robot

Potential energy of i-th link

$$\mathcal{P}_i = m_i g^T r_{c,i}$$

where $r_{c,i}$ is a CoM coordinates vector

$$\begin{vmatrix} r_{c,i} \\ 1 \end{vmatrix} = {}^0H_1(q_1)\,{}^1H_2(q_2)\cdots{}^{i-1}H_i(q_i) \begin{vmatrix} {}^i r_{c,i} \\ 1 \end{vmatrix} \quad (5)$$

Robot potential energy

$$\mathcal{P} = \sum_{i=1}^{n} \mathcal{P}_i = \sum_{i=1}^{n} m_i g^T r_{c,i}$$

For a serial kinematic chain

$$\mathcal{P} = \sum_{i=1}^{n} \mathcal{P}_i$$

$$\mathcal{P}_i = \mathcal{P}_i(q_j, j \le i)$$

Motion Equation

- Kinetic energy

$$\begin{aligned}\mathcal{K} &= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q)\right]\dot{q} \\ &= \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{k,j} m_{kj}(q)\dot{q}_k\dot{q}_j\end{aligned}$$

- for conservative generalized forces $\psi_k = -\frac{\partial \mathcal{P}}{\partial q_k} + \tau_k$
- system Lagrangian $\mathcal{L} = \mathcal{K} - \mathcal{P}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$$

$$\frac{d}{dt}\frac{\partial (\mathcal{K}-\mathcal{P})}{\partial \dot{q}_k} - \frac{\partial (\mathcal{K}-\mathcal{P})}{\partial q_k} = \tau_k$$

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial (\mathcal{K}-\mathcal{P})}{\partial q_k} = \tau_k$$

Motion Equation (contd.)

Equation structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial (\mathcal{K}-\mathcal{P})}{\partial q_k} = \tau_k, \qquad k = 1,\dots,n$$

1st term

$$\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right] = \sum_{j=1}^{n} m_{kj}\dot{q}_j$$

and

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{d}{dt}\left[\sum_{j=1}^{n} m_{kj}\dot{q}_j\right] &= \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{j=1}^{n}\tfrac{d}{dt}\left[m_{kj}(q)\right]\dot{q}_j \\ &= \sum_{j=1}^{n} m_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j\end{aligned}$$

Motion Equation (contd.)

2nd term

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}M(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}M(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P}$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}$$

Motion Equation (contd.)

2nd term

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}M(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}M(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P}$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}$$

Resulting relations

$$\sum_{j=1}^{n}m_{kj}\ddot{q}_j+\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial m_{kj}}{\partial q_i}+\frac{\partial m_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j$$
$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j+\frac{\partial}{\partial q_k}\mathcal{P}=\tau_k$$

Motion Equation (contd.)

2nd term

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}M(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}M(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P}$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}$$

Resulting relations

$$\sum_{j=1}^{n} m_{kj}(q)\ddot{q}_j+\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ijk}(q)\dot{q}_i\dot{q}_j+g_k(q)=\tau_k\,,\quad k=1,\dots,n$$

where $c_{ijk}=c_{jik}$ is a Christoffel symbol and

$$c_{ijk}(q)=\frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i}+\frac{\partial m_{ki}}{\partial q_j}-\frac{\partial m_{ij}}{\partial q_k}\right),\quad g_k(q)=\frac{\partial}{\partial q_k}\mathcal{P}$$

is a potential energy gradient

Motion Equation (contd.)

2nd term

$$\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}M(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}M(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P}$$
$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_k}\mathcal{P}$$

Resulting relations

$$\sum_{j=1}^{n} m_{kj}(q)\ddot{q}_j+\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ijk}(q)\dot{q}_i\dot{q}_j+g_k(q)=\tau_k\,,\quad k=1,\dots,n$$

in a vectorial form

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=\tau$$

with Coriolis and centrifugal forces $C(q)$, $c_{kj}=\sum_{i=1}^{n}c_{ijk}(q)\dot{q}_i$.

Accounting for Gear and Motor Dynamics

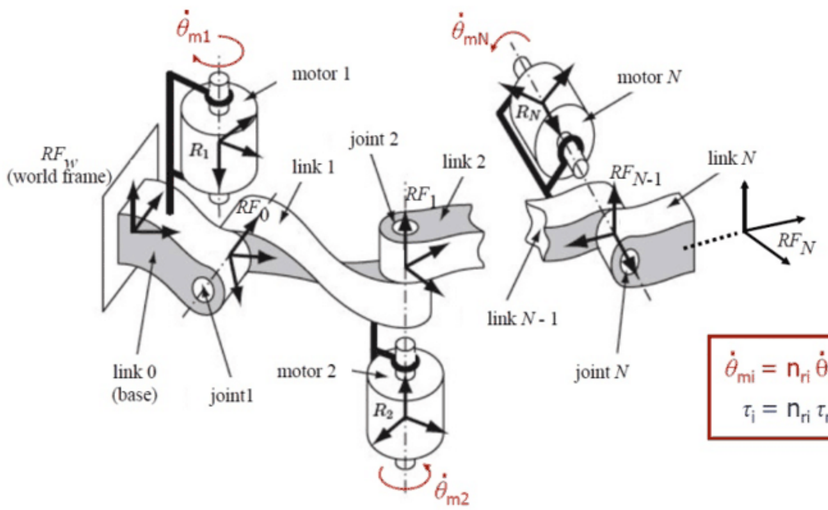
Assumptions

- drive is fixed to the link preceding the link it is moving
- motor and joint axis are coinciding

General considerations

- drive mass should be added to link mass
- drive rotor inertia should be taking into account when computing total kinetic energy
- gear ration should be taken into account when computing velocities and forces

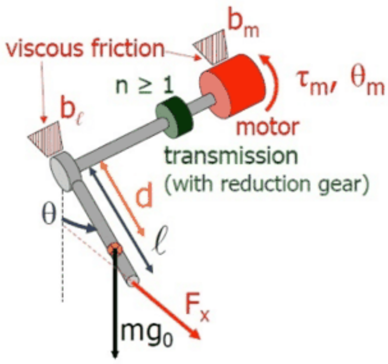
Motor Placement



$$\dot{\theta}_{mi} = n_{ri}\,\dot{\theta}_i$$
$$\tau_i = n_{ri}\,\tau_{mi}$$

©DeLuca

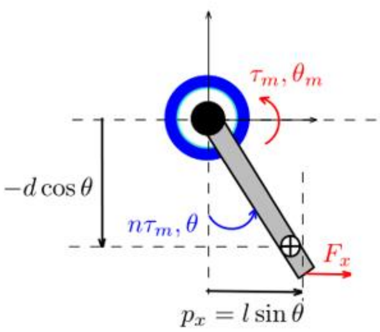
Pendulum with Gear



©DeLuca

I_l – link moment of inertia w.r.t. its CoM
m – link mass
d – distance from axis of rotation to link CoM
$\dot{\theta}$ – link rotation velocity (after gear)
$\dot{\theta}_m = n\dot{\theta}$ – motor rotation velocity (before gear)
n – gear ratio
I_m – drive moment of inertia w.r.t its axis of rotation

Kinetic Energy



Pendulum kinetic energy

$$\mathcal{K}_l = \frac{1}{2}\left(I_l + md^2\right)\dot{\theta}^2,$$

Drive kinetic energy

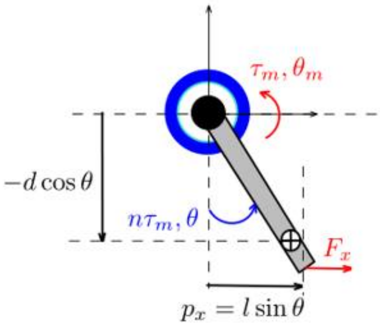
$$\mathcal{K}_m = \frac{1}{2}I_m\dot{\theta}_m^2,$$

Total kinetic energy

$$\mathcal{K} = \mathcal{K}_l + \mathcal{K}_m = \frac{1}{2}I\dot{\theta}^2,$$

where $I = I_l + md^2 + n^2 I_m$ is a total moment of inertia w.r.t. axis of rotation

Potential Energy and Lagrangian



Total potential energy

$$\mathcal{P} = \mathcal{P}_0 - mg_0 d\cos\theta.$$

System Lagrangian

$$\mathcal{L} = \frac{1}{2}I\dot{\theta}^2 + mg_0 d\cos\theta - \mathcal{P}_0.$$

Motion Equation

From the link side

$$I\ddot{\theta} + mg_0 d\sin\theta = \tau.$$

From the motor side

$$\frac{I}{n^2}\ddot{\theta}_m + \frac{m}{n}g_0 d\sin\frac{\theta_m}{n} = \tau_m - \left(\frac{k_{fl}}{n^2} + k_{fm}\right)\dot{\theta}_m + \frac{l}{n}\cos\frac{\theta_m}{n}F_x.$$

Friction Forces

General considerations

- is a dissipative force
- localized in joints
- static model captures major influence for relatively fast motion

$$\tau - n\tau_m - k_{fl}\dot{\theta} - nk_{fm}\dot{\theta}_m + \dot{p}_x F_x - n\tau_m - (k_{fl} + n^2 k_{fm})\dot{\theta} + l\cos\theta F_x,$$

where τ_m is drive torque before gear, k_{fm} and k_{fl} are viscous friction coefficients

- dynamic models are more accurate, but usually hard to identify

Flexible-Joints Robots

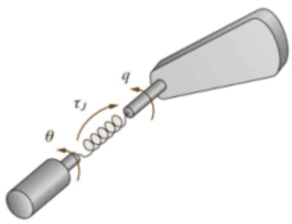


Figure 2: Flexible joint sketch

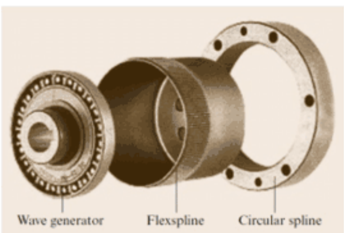


Figure 3: Harmonic drive

Flexible joints
Motor (input) and link (output) are connected by a flexible (deformable) element

- long shaft
- harmonic drive gearbox
- belts

Useful flexibility

1. physically (VSA, SEA)
2. on a software level

for

- safe pHRI
- explosive motions

Modeling Flexible Joints

Assumptions

1. flexibility is localized in joint
2. small deformations for linear spring model
3. symmetric drive shafts with CoM on the axis of rotation
4. drive is located before the link it is actuating

Modeling Flexible Joints

- introduce $2n$ generalized coordinates $q \in R^n$ for links and $\theta \in R^n$ for drives ($\theta_i = \theta_{mi}/r_i$, r_i is a gear ratio)
- add drive kinetic energy

$$\mathcal{K}_{mi} = \frac{1}{2}\mathcal{I}_m \dot{\theta}_{mi}^2 = \frac{1}{2}\mathcal{I}_m r_i^2 \dot{\theta}_i^2$$

$$\mathcal{K}_m = \sum_{i=1}^{n} \mathcal{K}_{mi} = \frac{1}{2}\dot{\theta}^T M_m \dot{\theta}$$

  M_m is a diagonal drive inertia matrix

- add potential energy of a deformed spring

$$\mathcal{P}_{ei} = \frac{1}{2}K_i(q_i - \theta_i)^2$$

$$\mathcal{P}_e = \sum_{i=1}^{n} \mathcal{P}_{ei} = \frac{1}{2}(q-\theta)^T K (q-\theta)$$

  K is a matrix of joint stiffness coefficients

Modeling Flexible Joints

Motion equation

$$M(q)\ddot{q} + c(q,\dot{q}) + G(q) + K(q-\theta) = 0,$$
$$M_m\ddot{\theta} + K(\theta - q) = \tau$$

Operational Space Formulation

- Configuration space

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = \tau$$

- Operational space

$$\Lambda\ddot{x}_e + \mu + \rho = F_e$$

Operational Space Formulation

- Configuration space

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = \tau$$

- Operational space

$$\Lambda \ddot{x}_e + \mu + \rho = F_e$$

- projecting joint forces/torques to end-effector forces

$$\tau = J_e^T F_e$$

- kinematic relations

$$\dot{x}_e = J_e \dot{q} \Rightarrow \ddot{x}_e = J_e \ddot{q}_e + \dot{J}_e \dot{q}_e$$

$$\ddot{x}_e = J_e M^{-1}\left(J_e^T F_e - (c(q,\dot{q}) + g(q))\right) + \dot{J}_e \dot{q}_e \Rightarrow$$
$$\ddot{x}_e + J_e M^{-1}(c(q,\dot{q}) + g(q)) - \dot{J}_e \dot{q}_e = J_e M^{-1} J_e^T F_e$$

- operational-space model

$$\Lambda = \left(J_e M^{-1} J_e^T\right)^{-1} \qquad \mu = \Lambda J_e M^{-1} c(q,\dot{q}) - \Lambda \dot{J}_e \dot{q}_e \qquad \rho = \Lambda J_e M^{-1} g(q)$$

Digital control systems

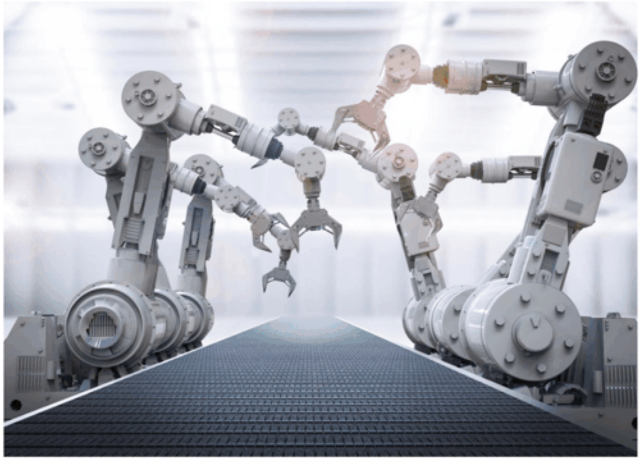
Digital and microcontroller devices

Digital and Microcontroller Devices

Vlasov Sergei

Robots, what is it?

Robots, what is it?

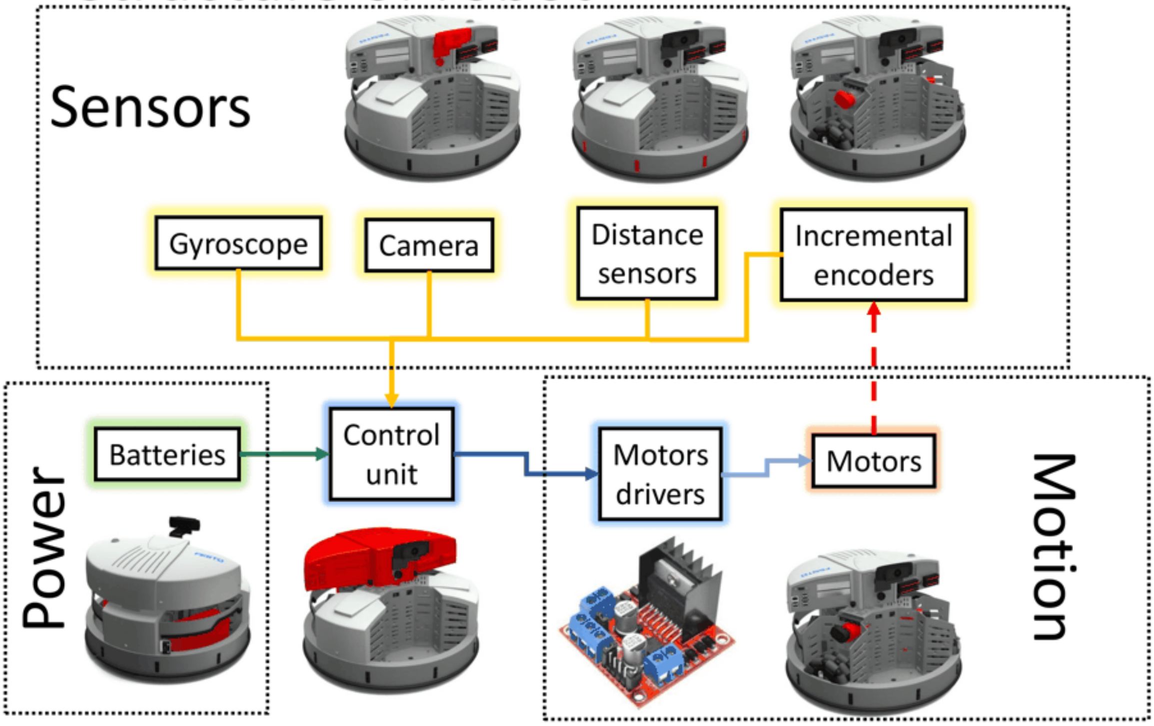


Structure of robot

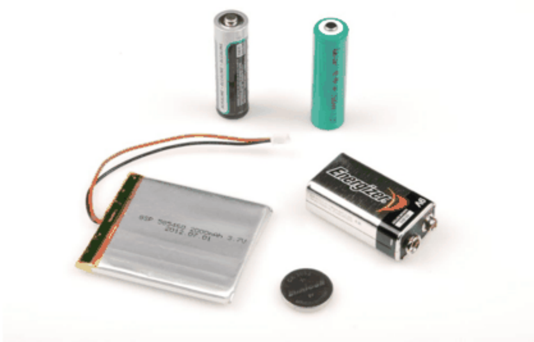
Hardware

Control	Drive systems	Sensors	Interfaces	Supply
→ Power switch	→ Omnidrive	→ Bumper	→ WLAN	→ Batteries
→ Control unit	→ Motors	→ Distance sensors	→ I/O-Interfaces	→ Power supply unit
→ Embedded PC	→ Incremental encoder	→ Gyroscope	→ Motor/encoder	→ Charging electronics
→ Microcontroller	→ Gear units	→ Camera	→ USB	→ Pedestal
→ Reset button	→ Wheels	→ Opto-electronic sensors	→ PCI Express	
		→ Inductive sensors	→ Ethernet	
			→ VGA	

Structure of robot



Power supply



- Primary Batteries
- Secondary Batteries
    - Lithium (Li-ion, Li-pol)
    - Nickel Cadmium (Ni-Cd)
    - Nickel-Metal Hydride (Ni-MH)
    - Lead-Acid

Schematic symbols

Single cell   Multi-cell

Power supply

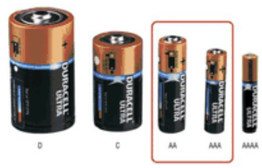
Terminology

- **Capacity** - Batteries have different ratings for the amount of power a given battery can store. When a battery is fully charged, the capacity is the amount of power it contains. Batteries of the same type will often be rated by the amount of current they can output over time. For example, there are 1000mAh (milli-Amp Hour) and 2000mAh batteries.
- **Nominal Cell Voltage** - The average voltage a cell outputs when charged. The nominal voltage of a battery depends on the chemical reaction behind it. A lead-acid car battery will output 12V. A lithium coin cell battery will output 3V.
- The key word here is "nominal", the actual measured voltage on a battery will decrease as it discharges. A fully charged LiPo battery will produce about 4.23V, while when discharged its voltage may be closer to 2.7V.
- **Shape** - Batteries come in many sizes and shapes. The term 'AA' references a specific shape and style of a cell. There are a large variety.

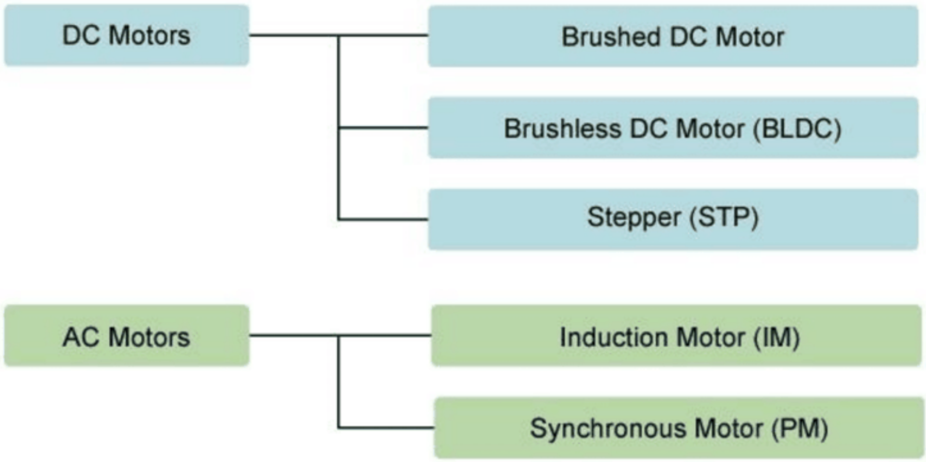
Power supply

Common batteries, their chemistry, and their nominal voltage

Battery Shape	Chemistry	Nominal Voltage	Rechargeable?
AA, AAA, C, and D	Alkaline or Zinc-carbon	1.5V	No
9V	Alkaline or Zinc-carbon	9V	No
Coin Cell	Lithium	3V	No
Silver Flat Pack	Lithium Polymer (LiPo)	3.7V	Yes
AA, AAA, C, D (Rechargeable)	NiMH or NiCd	1.2V	Yes
Car Battery	Six-cell lead acid	12.6V	Yes

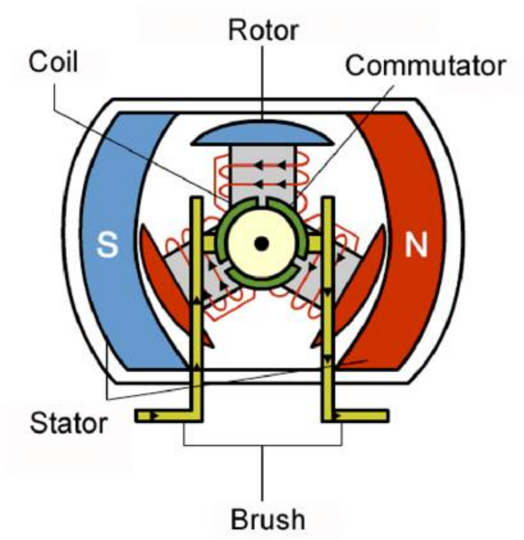


Motion – motors



Brushed DC electric motor

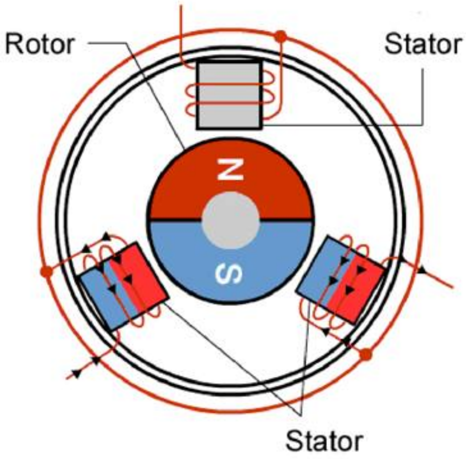
Fixed brushes supply electric energy to the rotating commutator. As the commutator rotates, it continually flips the direction of the current into the coils, reversing the coil polarities so that the coils maintain rightward rotation. The commutator rotates because it is attached to the rotor on which the coils are mounted.



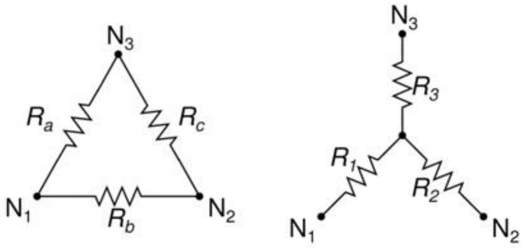
Schematic symbols

Brushless DC electric motor

Since the rotor is a permanent magnet, it needs no current, eliminating the need for brushes and commutator. Current to the fixed coils is controlled from the outside.

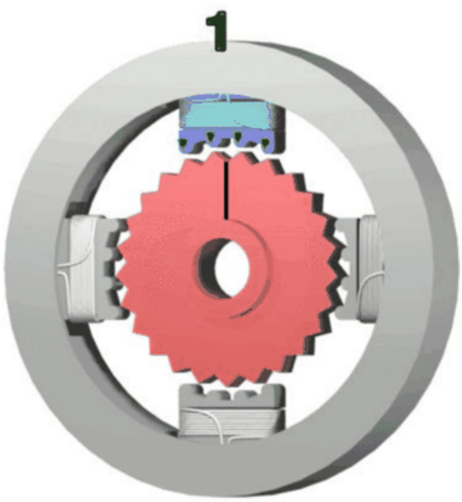


Schematic for delta and wye winding styles. (This image does not illustrate the motor's inductive and generator-like properties)

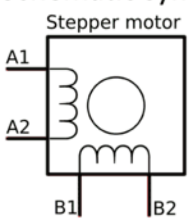


Stepper motor

A stepper motor, also known as step motor or stepping motor, is a brushless DC electric motor that divides a full rotation into a number of equal steps. The motor's position can then be commanded to move and hold at one of these steps without any position sensor for feedback (an open-loop controller), as long as the motor is carefully sized to the application in respect to torque and speed.

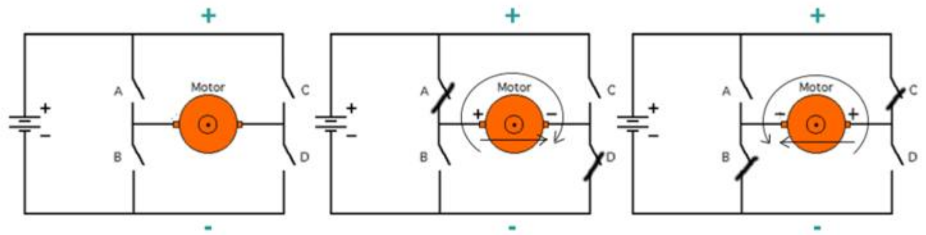


Schematic symbols

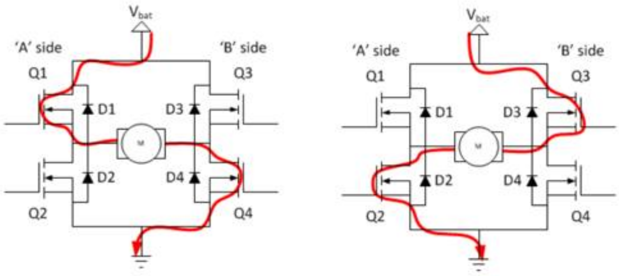


Controlling Brushed DC Motors

Rotation in different directions

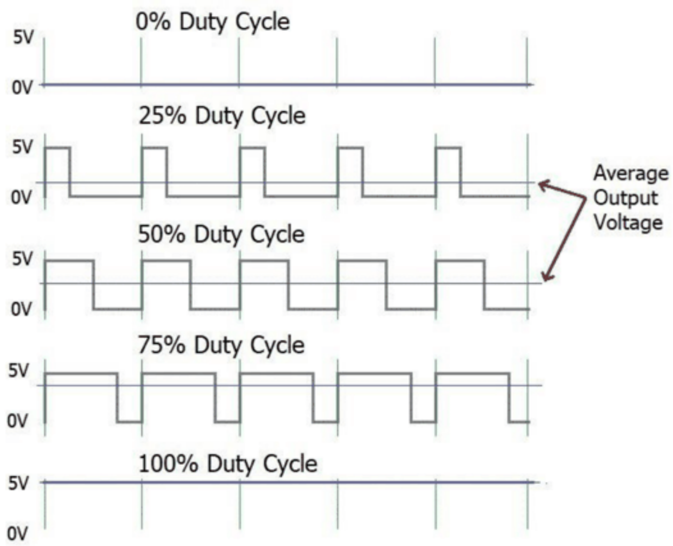


Schematic for microcontroller use

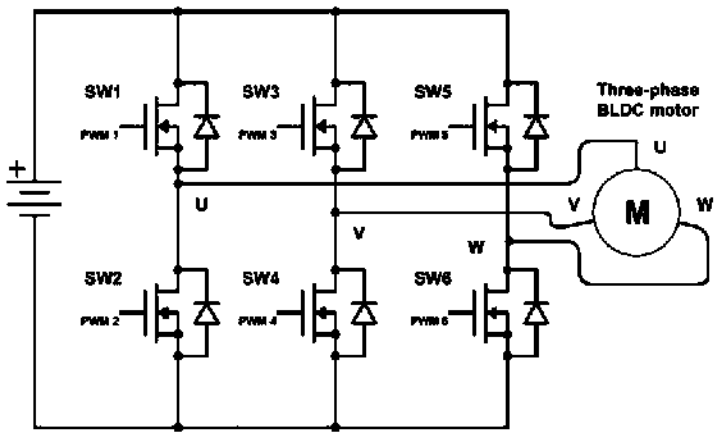


Controlling Brushed DC Motors

Control speed by PWM (Pulse-Wide Modulation)



Controlling Brushless DC Motors

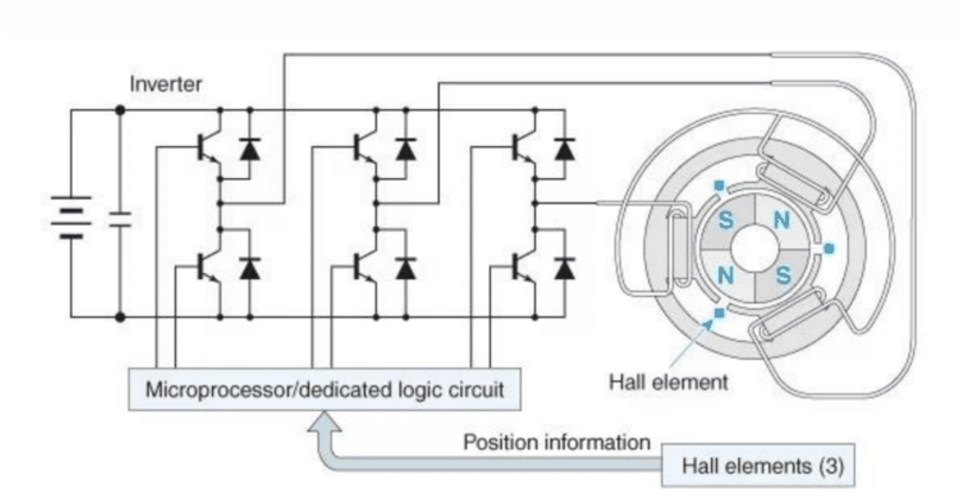


Controlling Brushless DC Motors

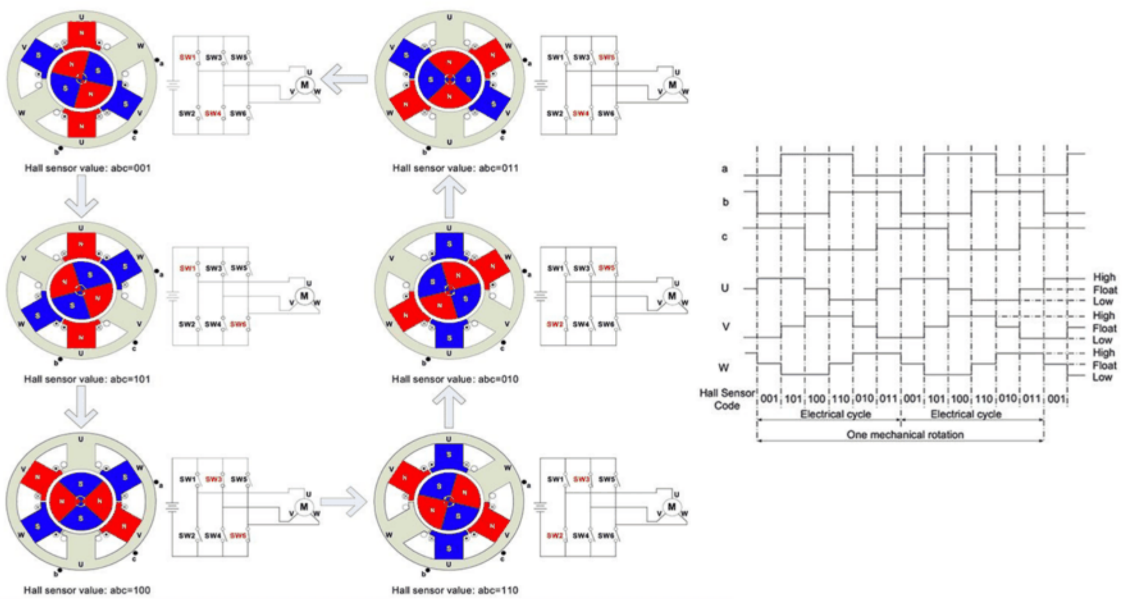
Sensored vs. sensorless

- Two technologies offer a solution for positional feedback. The first and most common uses three Hall-effect sensors embedded in the stator and arranged at equal intervals, typically 60° or 120°. A second, 'sensorless' control technology comes into its own for BLDC motors that require minimal electrical connections.

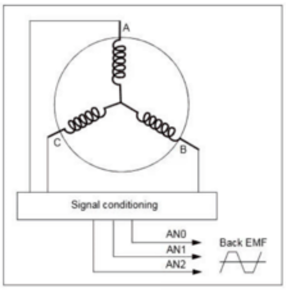
Controlling Brushless DC Motors with sensors

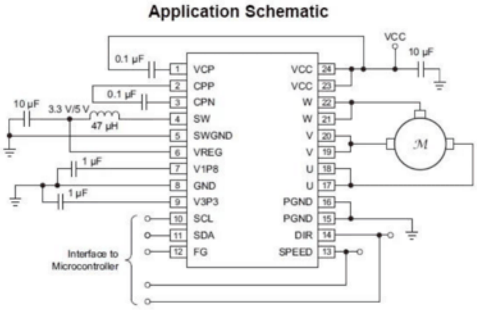


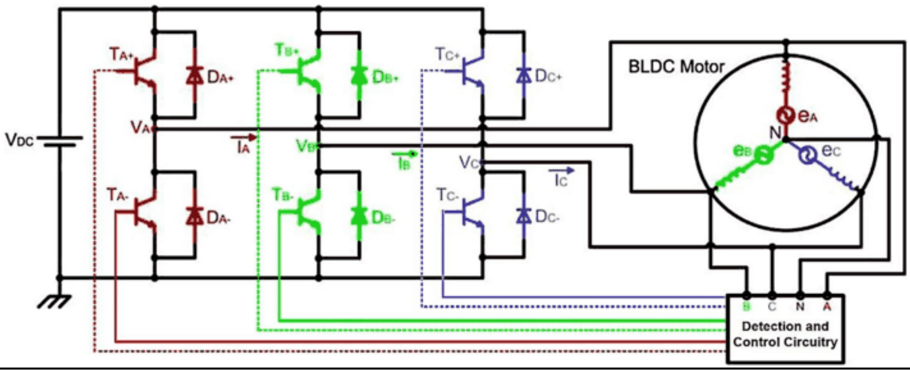
Controlling Brushless DC Motors with sensors



Controlling Brushless Sensorless DC Motors

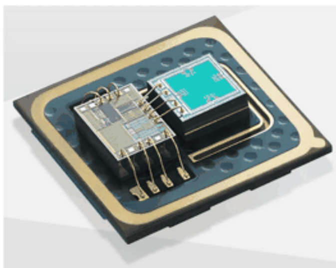






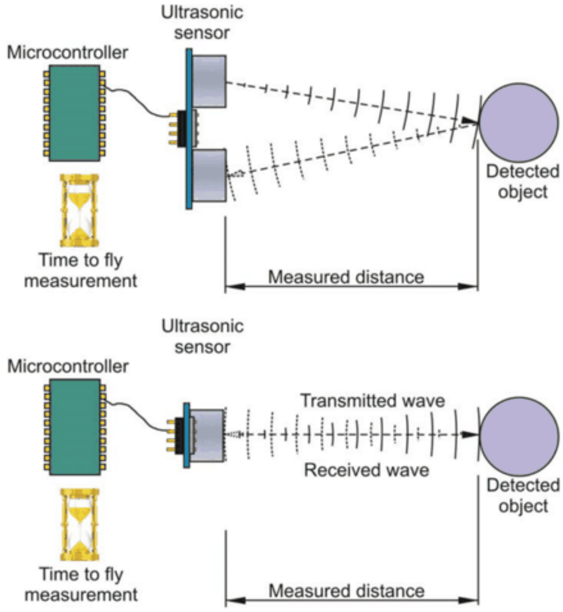
Sensors

- Distance
- Position
- Velocity
- Temperature



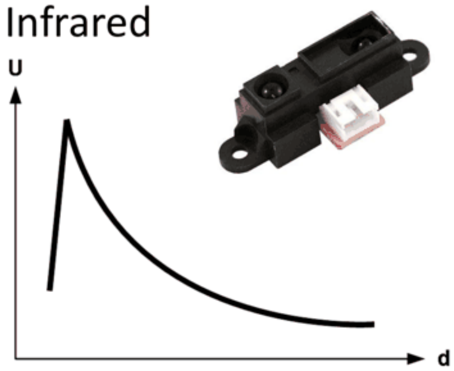
Distance sensors

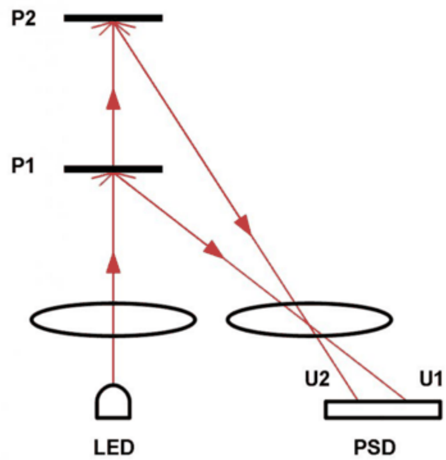
- Ultrasonic



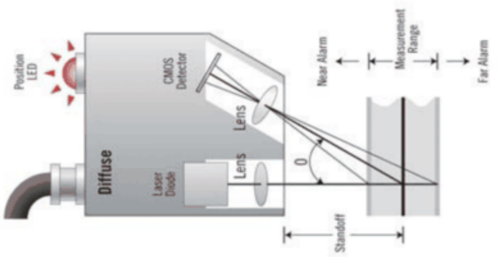
Distance sensors

- Infrared





- Laser

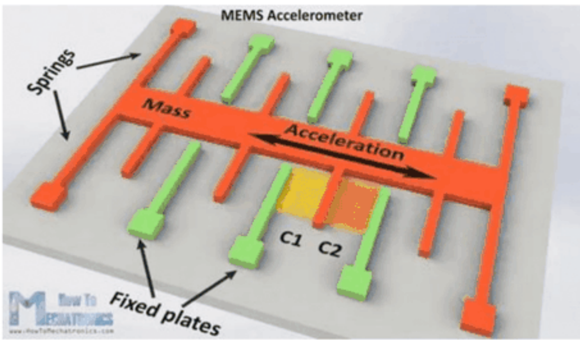


Position/Velocity sensors

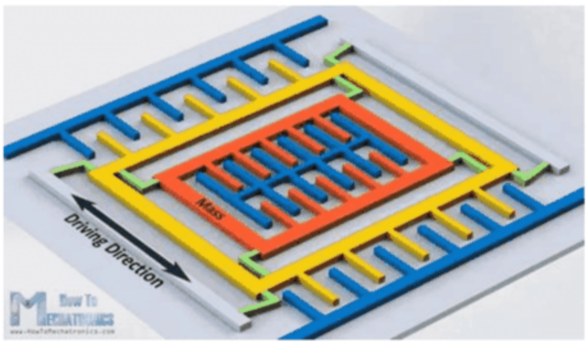
- MEMS (Micro Electro Mechanical Systems)
  - Accelerometer
  - Gyroscope
  - Magnetometer
- Encoder
- Potentiometer

MEMS sensors

- Accelerometer

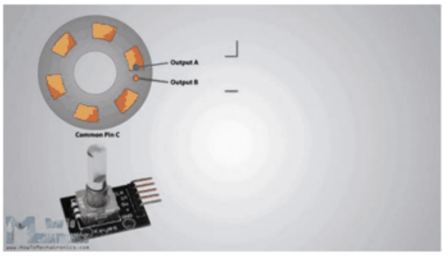


- Gyroscope

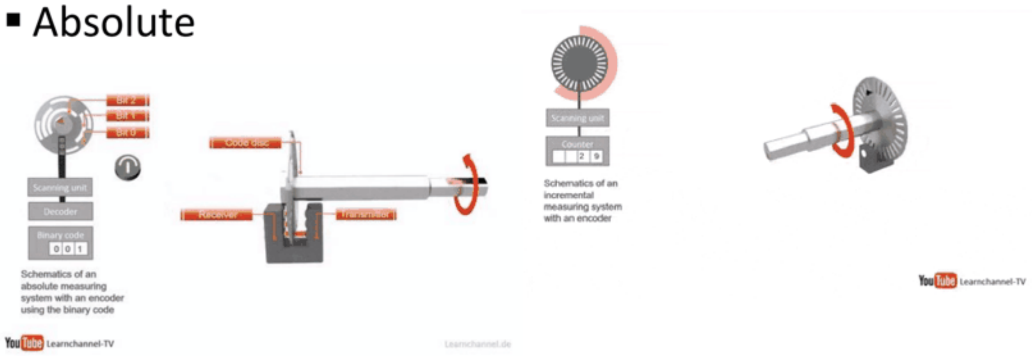


Encoders

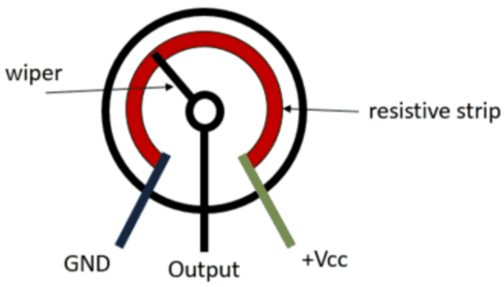
- Mechanical
- Optical

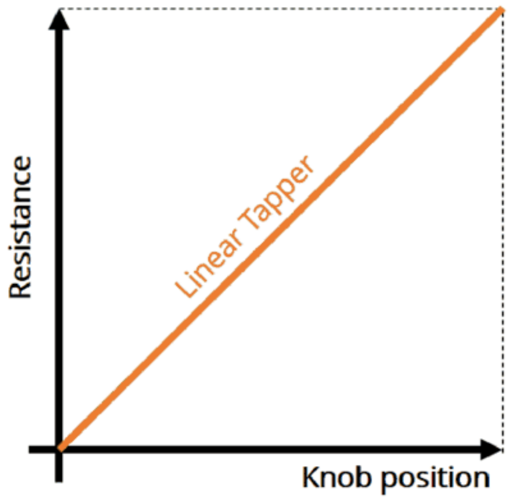


- Incremental
- Absolute



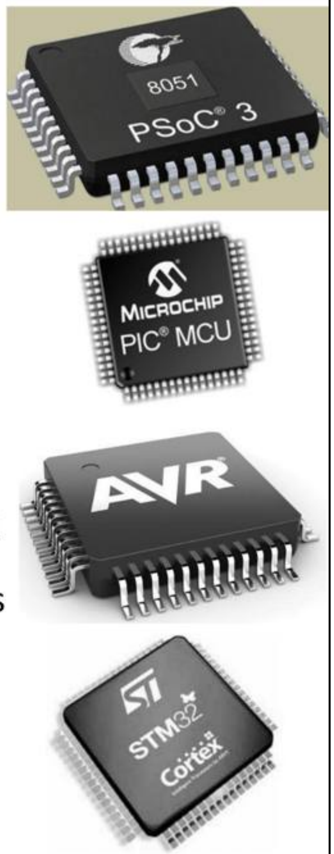
Potentiometer



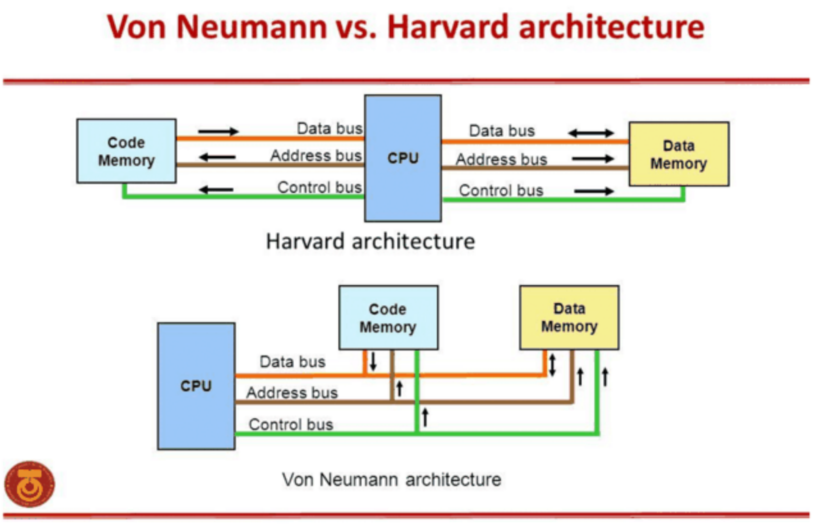


Microcontrollers

A micro-controller can be comparable to a little stand alone computer; it is an extremely powerful device, which is able of executing a series of pre-programmed tasks and interacting with extra hardware devices. Being packed in a tiny integrated circuit (IC) whose size and weight is regularly negligible, it is becoming the perfect controller for as robots or any machines required some type of intelligent automation. A single microcontroller can be enough to manage a small mobile robot, an automatic washer machine or a security system. Several microcontrollers contains a memory to store the program to be executed, and a lot of input/output lines that can be a used to act jointly with other devices, like reading the state of a sensor or controlling a motor.

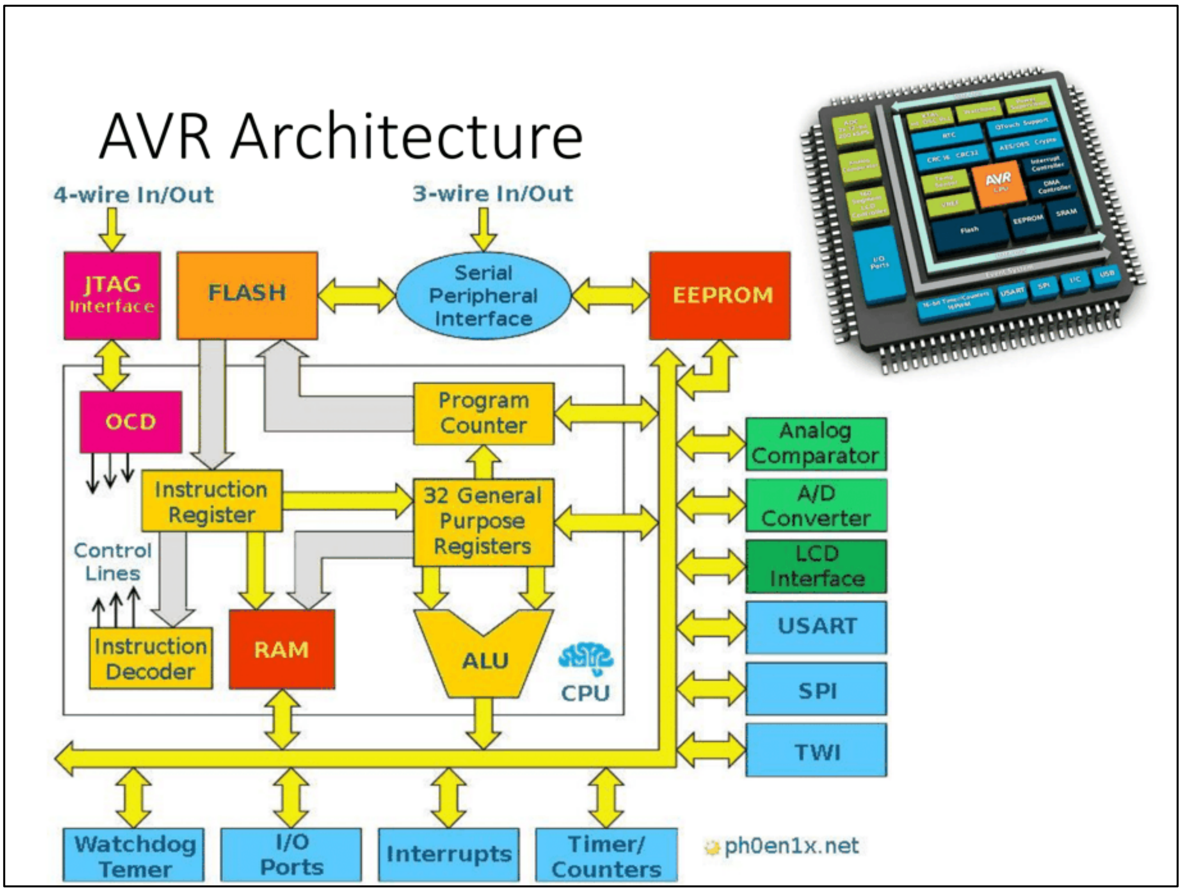


Microcontroller's architecture



Microcontrollers

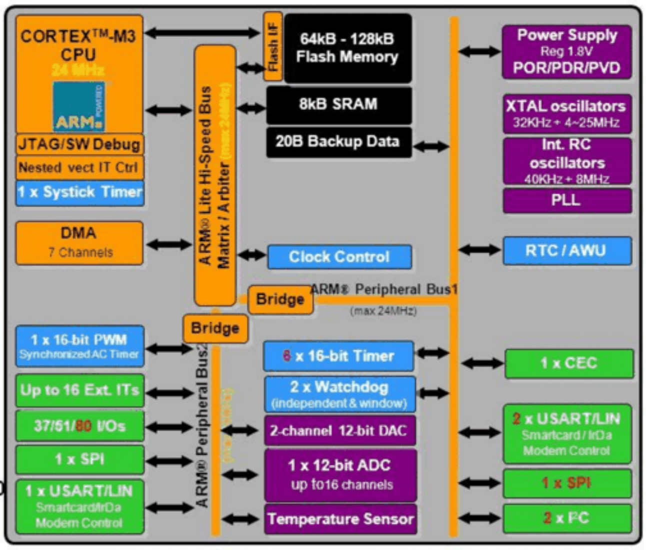
	8051	PIC	AVR	ARM
Bus width	8-bit for standard core	8/16/32-bit	8/32-bit	32-bit mostly also available in 64-bit
Communication Protocols	UART, USART,SPI,I2C	PIC, UART, USART, LIN, CAN, Ethernet, SPI, I2S	UART, USART, SPI, I2C, (special purpose AVR support CAN, USB, Ethernet)	UART, USART, LIN, I2C, SPI, CAN, USB, Ethernet, I2S, DSP, SAI (serial audio interface), IrDA
Speed	12 Clock/instruction cycle	4 Clock/instruction cycle	1 clock/ instruction cycle	1 clock/ instruction cycle
Memory	ROM, SRAM, FLASH	SRAM, FLASH	Flash, SRAM, EEPROM	Flash, SDRAM, EEPROM
ISA	CLSC	Some feature of RISC	RISC	RISC
Memory Architecture	Von Neumann architecture	Harvard architecture	Modified	Modified Harvard architecture
Power Consumption	Average	Low	Low	Low
Families	8051 variants	PIC16,PIC17, PIC18, PIC24, PIC32	Tiny, Atmega, Xmega, special purpose AVR	ARMv4,5,6,7 and series
Community	Vast	Very Good	Very Good	Vast
Manufacturer	NXP, Atmel, Silicon Labs, Dallas, Cyprus, Infineon, etc.	Microchip Average	Atmel	Apple, Nvidia, Qualcomm, Samsung Electronics, and TI etc.
Cost (as compared to features provide)	Very Low	Average	Average	Low
Other Feature	Known for its Standard	Cheap	Cheap, effective	High speed operation Vast
Popular Microcontrollers	AT89C51, P89v51, etc.	PIC18fXX8, PIC16f88X, PIC32MXX	Atmega8, 16, 32, Arduino Community	LPC2148, ARM Cortex-M0 to ARM Cortex-M7, etc.



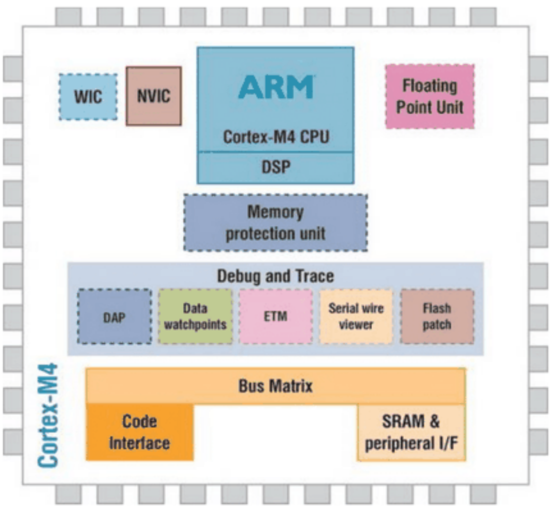
ARM – STM32 Architecture

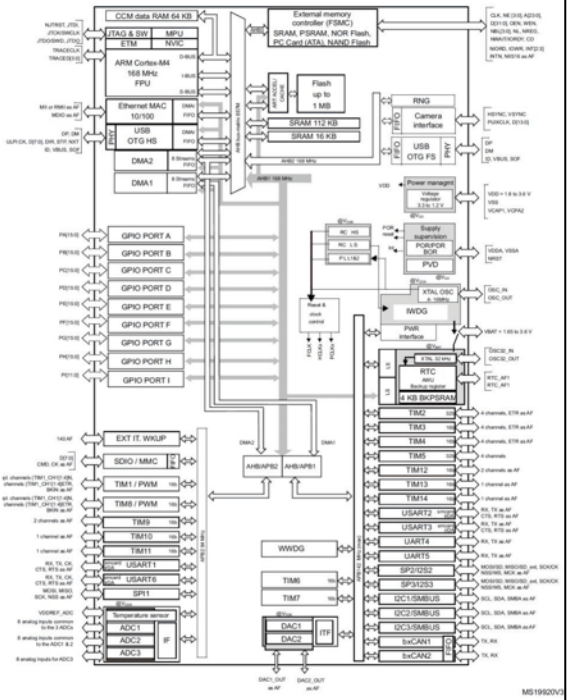
STM32 Value line 64K-128KBytes block diagram

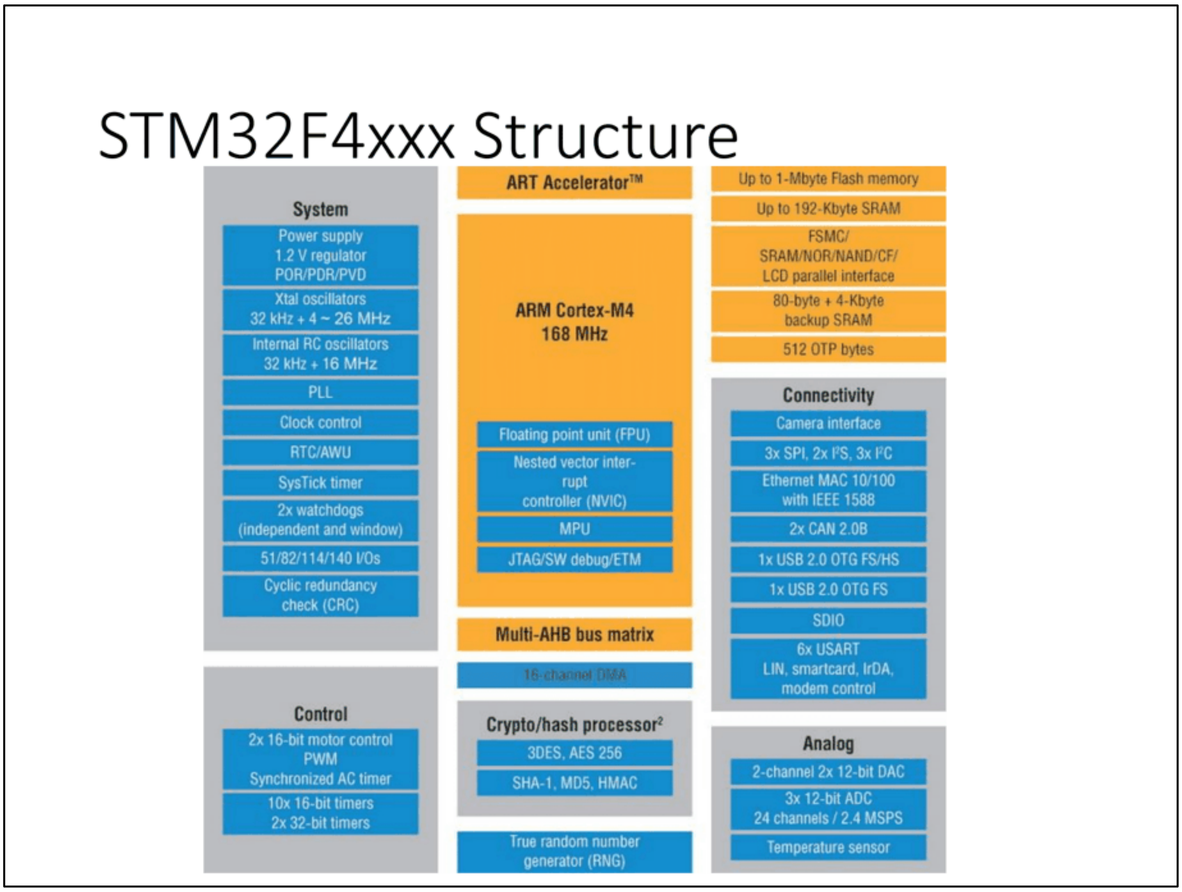
- **Core and operating conditions**
  - ARM® Cortex™-M3 1.25 DMIPS/MHz up to 24 MHz
  - 2.0 V to 3.6 V range
  - -40 to +105 °C
- **Rich connectivity**
  - 8 communications peripherals
- **Advanced analog**
  - 12-bit1.2 µs conversion time ADC
  - Dual channel 12-bit DAC
- **Enhanced control**
  - 16-bit motor control timer
  - 6x 16-bit PWM timers
- **LQFP48, LQFP/BGA64, LQFP100**

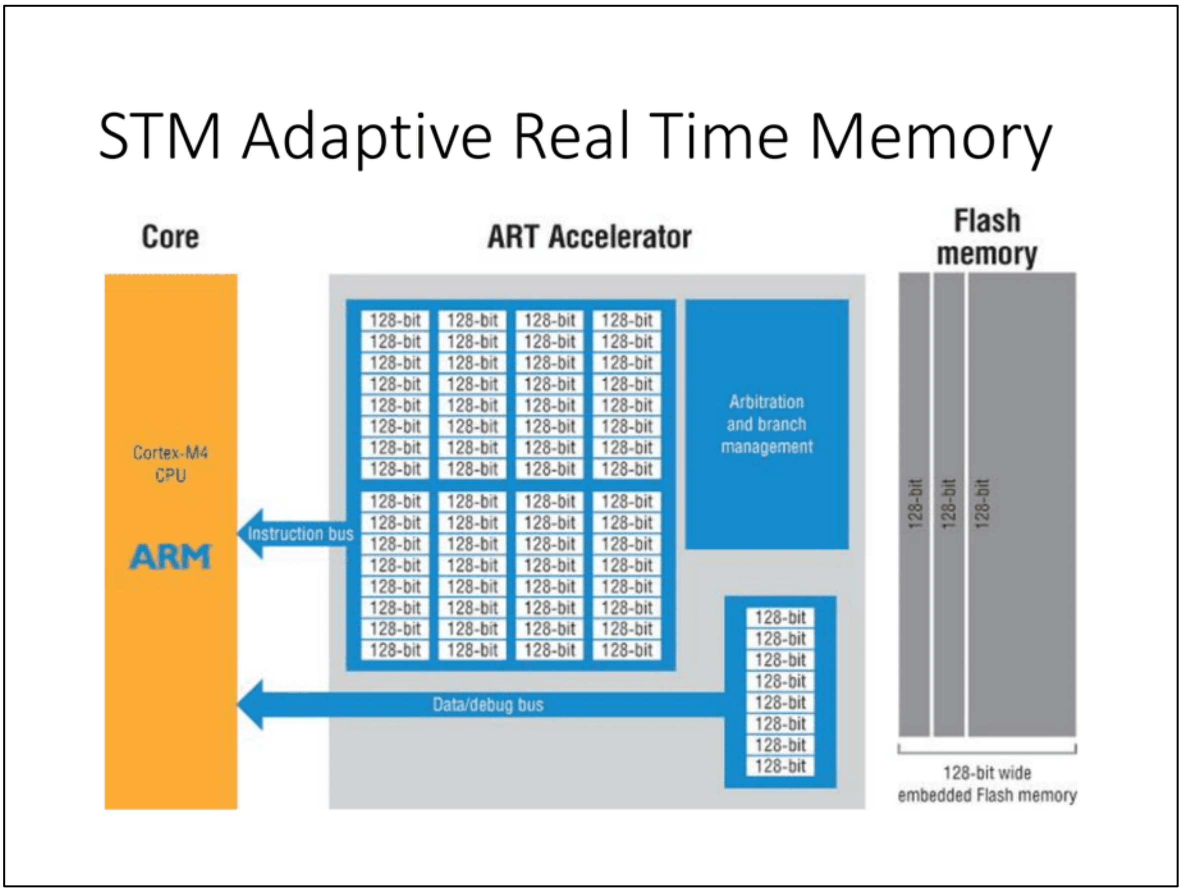


STM32F40xxx architecture

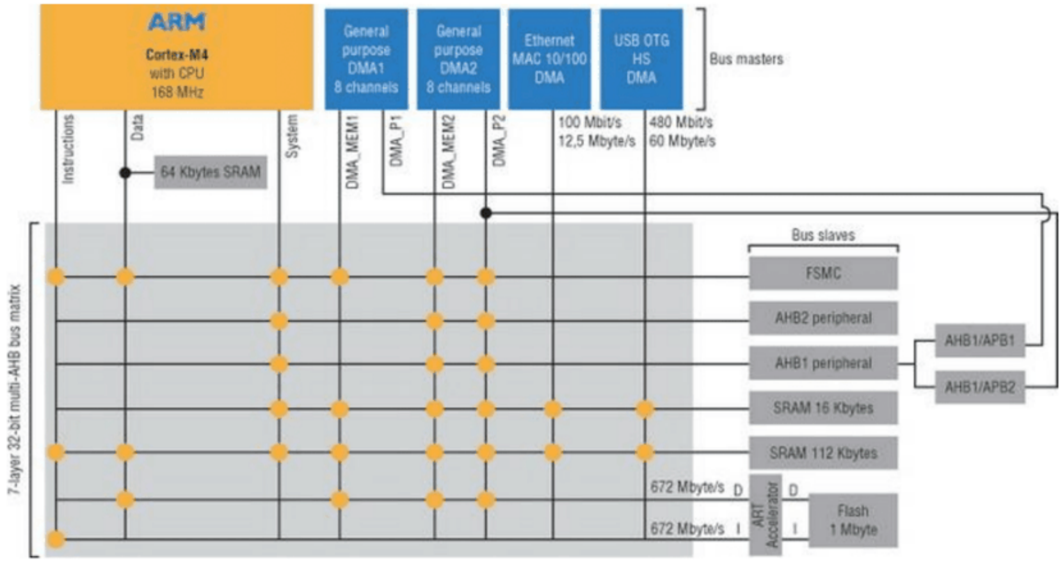




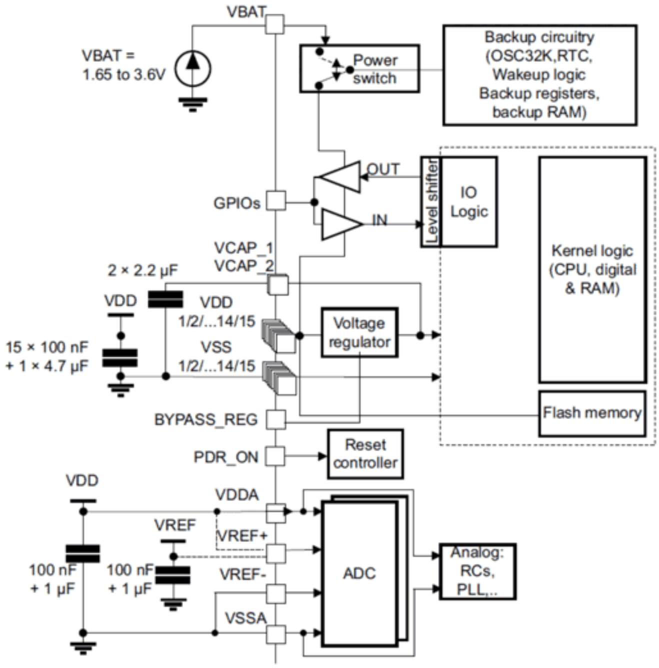

STM32F4xxx Structure
System
Power supply
1.2 V regulator
POR/PDR/PVD
Xtal oscillators
32 kHz + 4 ~ 26 MHz
Internal RC oscillators
32 kHz + 16 MHz
PLL
Clock control
RTC/AWU
SysTick timer
2x watchdogs
(independent and window)
51/82/114/140 I/Os
Cyclic redundancy
check (CRC)
Control
2x 16-bit motor control
PWM
Synchronized AC timer
10x 16-bit timers
2x 32-bit timers
ART Accelerator™
ARM Cortex-M4
168 MHz
Floating point unit (FPU)
Nested vector inter-
rupt
controller (NVIC)
MPU
JTAG/SW debug/ETM
Multi-AHB bus matrix
16-channel DMA
Crypto/hash processor²
3DES, AES 256
SHA-1, MD5, HMAC
True random number
generator (RNG)
Up to 1-Mbyte Flash memory
Up to 192-Kbyte SRAM
FSMC/
SRAM/NOR/NAND/CF/
LCD parallel interface
80-byte + 4-Kbyte
backup SRAM
512 OTP bytes
Connectivity
Camera interface
3x SPI, 2x I²S, 3x I²C
Ethernet MAC 10/100
with IEEE 1588
2x CAN 2.0B
1x USB 2.0 OTG FS/HS
1x USB 2.0 OTG FS
SDIO
6x USART
LIN, smartcard, IrDA,
modem control
Analog
2-channel 2x 12-bit DAC
3x 12-bit ADC
24 channels / 2.4 MSPS
Temperature sensor


STM Adaptive Real Time Memory
Core
ART Accelerator
Flash
memory
Cortex-M4
CPU
ARM
Instruction bus
Data/debug bus
128-bit
Arbitration
and branch
management
128-bit wide
embedded Flash memory

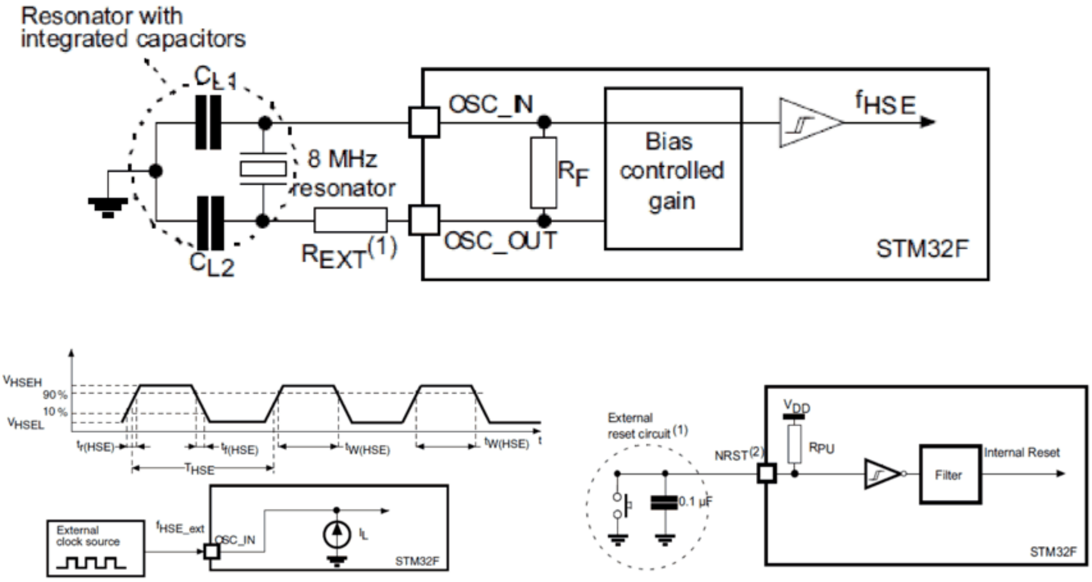
STM Bus matrix



Power supply scheme



External clocking and reset circuit



Actuators and mobile robots control
Mathematical model of DC motor

Control and modeling of mobile robots
Mathematical model of DC motor

Alexander A. Kapitonov

Constructon of DC motor

Figure 1. DC motor assembled.

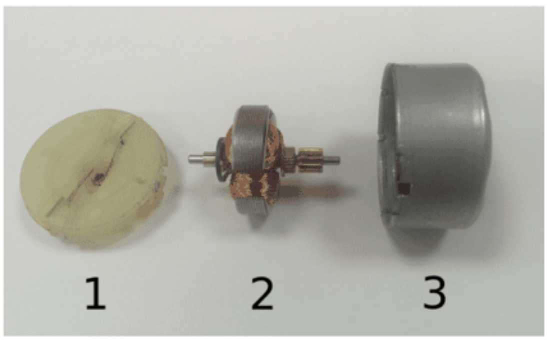


Figure 2. DC motor disassembled: 1 — cap, 2 — rotor, 3 — stator.

Constructon of DC motor

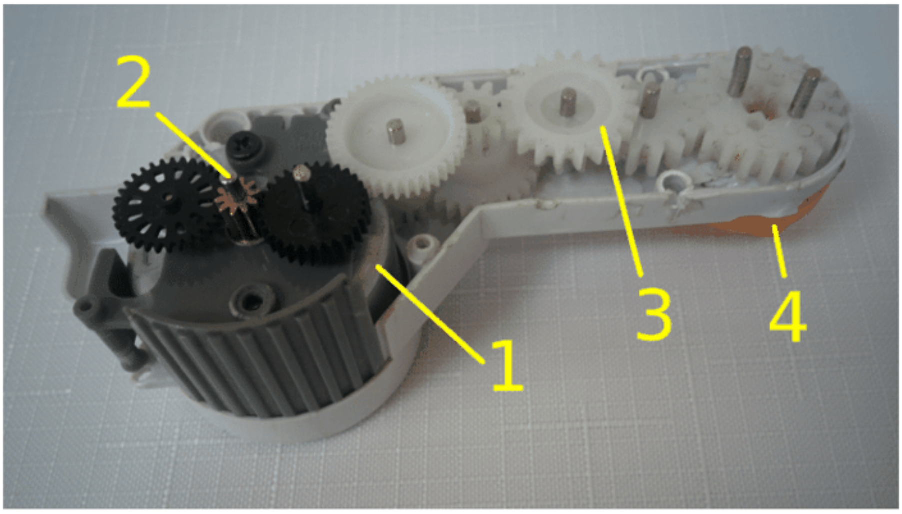


Figure 3. NXT motor disassembled: 1 — DC motor 2 — it's shaft, 3 — reducer, 4 — NXT motor external shaft; (chip with encoder isn't shown).

Mathematical model

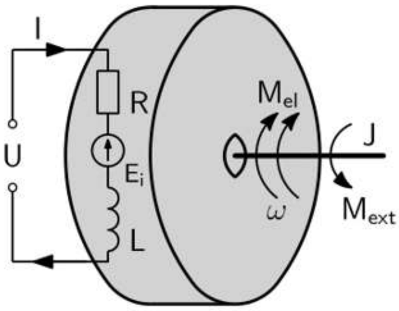


Figure 4. Physical scheme of DC motor.

$$\begin{cases} M_{el} - M_{ext} = J\dot{\omega}, \\ U = RI + E_i + L\dot{I}, \end{cases} \qquad (1)$$

$$M_{el} = k_m I, \qquad (2)$$

$$E_i = k_e \omega, \qquad (3)$$

$$\begin{cases} k_m I - M_{ext} = J\dot{\omega}, \\ U = RI + k_e\omega + L\dot{I}, \end{cases} \qquad (4)$$

where M_{el} — motor torque; M_{ext} — torque of external forces; J — total moment of inertia of the rotor and reducer's gears; ω — rotor speed; U — motor supply voltage; R, L — resistance and an inductance of rotor's wires; I — the current flowing through the latter; E_i — EMF which appeared in rotor's wires due to its rotation in magnetic field of stator's magnets; k_m, k_e — torque and back EMF motor constants.

Mathematical model

If $L \approx 0$ H, then

$$I = \frac{1}{R}U - \frac{k_e}{R}\omega, \tag{5}$$

therefore

$$k_m\left(\frac{1}{R}U - \frac{k_e}{R}\omega\right) - M_{ext} = J\dot{\omega}, \tag{6}$$

hence

$$\frac{JR}{k_m k_e}\dot{\omega} + \omega - \frac{1}{k_e}U - \frac{R}{k_m k_e}M_{ext}, \tag{7}$$

$$T_m\,\dot{\omega} + \omega = \frac{1}{k_e}U - \frac{T_m}{J}M_{ext}, \tag{8}$$

where $T_m = \dfrac{JR}{k_m k_e}$ is a motor mechanical constant.

Mathematical model

Also we can get differential equation which contains I, not ω:

1. differentiating (5):

$$\dot{\omega} = \frac{1}{k_e}\dot{U} - \frac{R}{ke}\dot{I} \tag{9}$$

2. putting (9) to the first equation from (4):

$$k_m I - M_{ext} = J\left(\frac{1}{k_e}\dot{U} - \frac{R}{ke}\dot{I}\right) \tag{10}$$

3. transforming (10):

$$\frac{JR}{k_m k_e}\dot{I} + I = \frac{J}{k_m k_e}\dot{U} + \frac{1}{k_m}M_{ext}, \tag{11}$$

$$T_m\dot{I} + I - \frac{T_m}{R}\dot{U} + \frac{1}{k_m}M_{ext}. \tag{12}$$

Mathematical model

For a situation when

$$\begin{cases} U = const, \\ M_{ext} = 0 \ N \cdot m, \end{cases} \tag{13}$$

and

$$\begin{cases} \theta(0) = 0, \\ \omega(0) = 0 \ s, {}^{1} \end{cases} \tag{14}$$

where θ — angle of rotor's rotation ($\dot{\theta} = \omega$), next expressions for $\omega(t)$ and $\theta(t)$ can be obtained from (8):

$$\omega(t) = \omega_{nls}\left(1 - \exp\left(\frac{t}{T_m}\right)\right), \tag{15}$$

$$\theta(t) = \omega_{nls}t - \omega_{nls}T_m + \omega_{nls}T_m \exp\left(\frac{t}{T_m}\right), \tag{16}$$

where $\omega_{nls} = U/k_e$ — no-load speed of the rotor.

Mathematical model

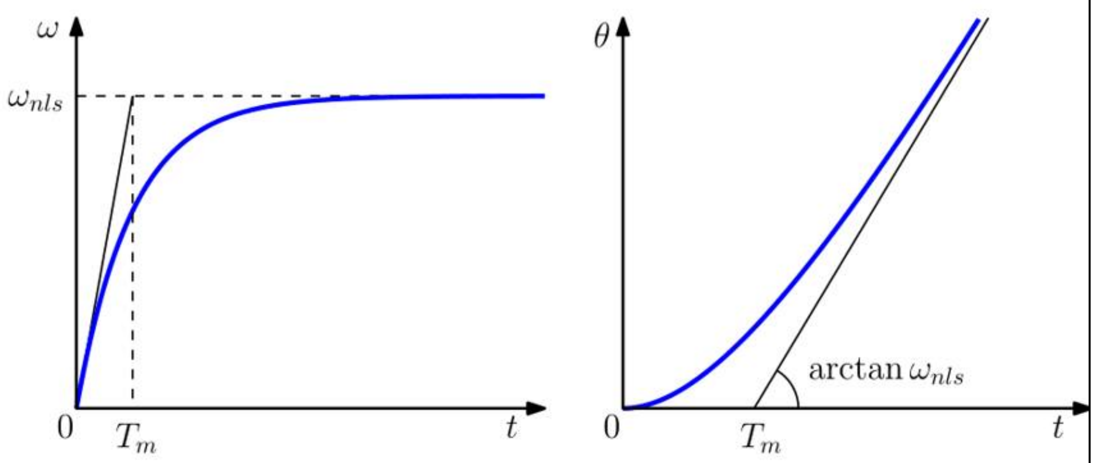


Figure 5. Graphs of $\omega(t)$ and $\theta(t)$ from (15) and (16) in case $\omega_{nls} > 0$.

Description of experiment

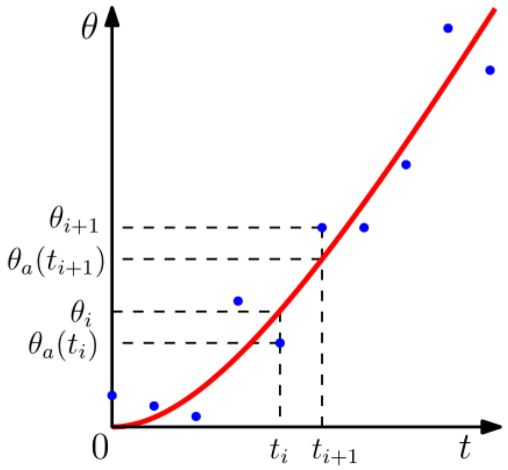


Figure 6. Approximation curve.

Least squares method:

find values for ω_{nls} and T_m such that the sum S:

$$S = \sum_{j=1}^{N} \left(\theta_a(t_j) - \theta_j\right)^2 \qquad (17)$$

would have a minimal possible value.

There

N — number of pairs (t_j, θ_j) which were recorded during the experiment,

$\theta_a(t_j)$ — value of (16) when $t = t_j$.

Modeling scheme of DC motor in Scilab

Control and modeling of mobile robots

Modeling scheme of DC motor in Scilab

Alexander A. Kapitonov

Scilab

Scilab is free and open source software for numerical computation providing a powerful computing environment for engineering and scientific applications.[1]

Figure 1. Scilab logo.

At this course we will use Scilab's:

- Xcos — hybrid dynamic systems modeler and simulator;
- some mathematical algorithms.

[1] Logo and some text on this slide were taken from www.scilab.org.

System modeling

Figure 2 demonstrates example of modeling scheme for device which is desribed by this system of equations:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \\ u = K(x_d - x) + Lx_g. \end{cases} \tag{1}$$

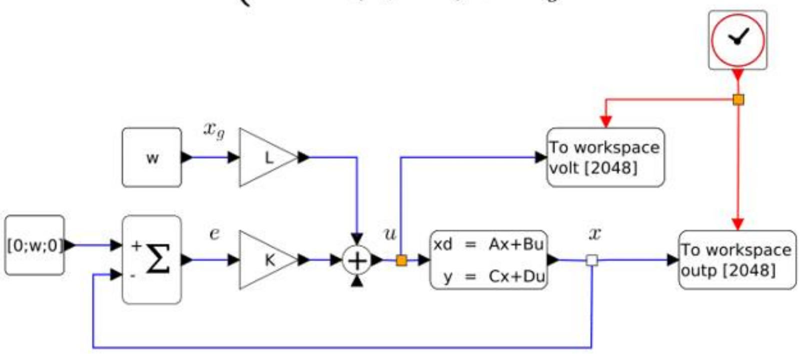
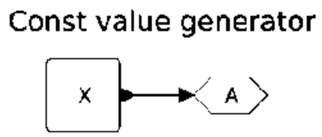
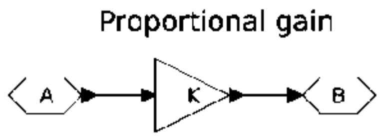


Figure 2. Example of modeling scheme.

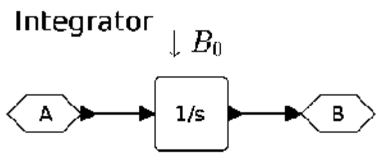
System modeling



$A = X = const$



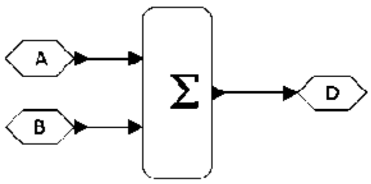
$B = K \cdot A$



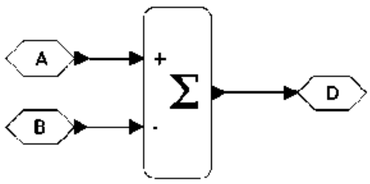
$$B = \int A\,dt + B_0$$

Figure 3. Some standard blocks.

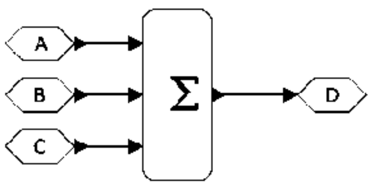
System modeling



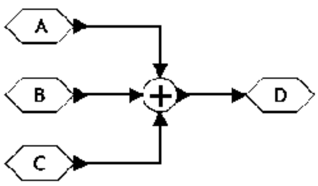
$D = A + B$



$D = A - B$



$D = A + B + C$



$D = A + B + C$

Figure 4. Summator block.

System modeling

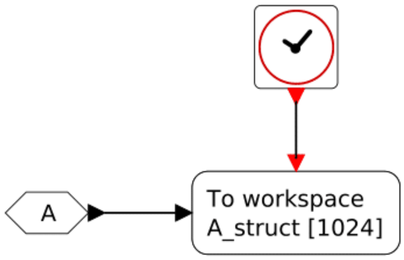


Figure 5. Some service blocks.

This subscheme saves values of A and appropriate moments of time into two matrices: `A_struct.values` and `A_struct.time`.

System modeling

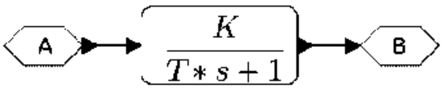


Figure 6. Transfer function block.

$$T \cdot \dot{B}(t) + B(t) = K \cdot A(t), \quad (2)$$

$$\mathcal{L}\{T \cdot \dot{B}(t) + B(t)\} = \mathcal{L}\{K \cdot A(t)\}, \quad (3)$$

$$T \cdot s \cdot B(s) - B(s) = K \cdot A(s), \quad (4)$$

$$\frac{B(s)}{A(s)} = \frac{K}{Ts+1}, \quad (5)$$

where $L\{\,\}$ — Laplace transform.

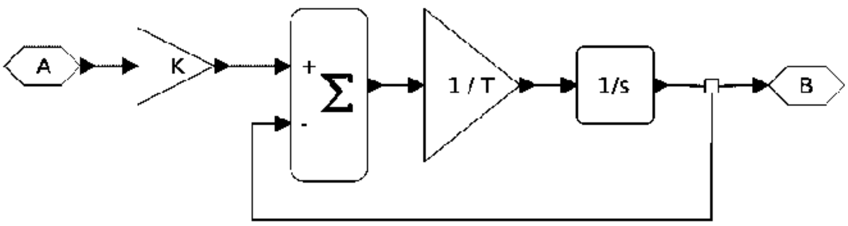


Figure 7. An equivalent sheme.

Modeling scheme of DC motor

Model of DC motor is described by two diffential equations:

$$\begin{cases} T_m\dot{\omega} + \omega = \frac{1}{k_e}U - \frac{T_m}{J}M_{ext}, \\ T_m\dot{I} + I = \frac{T_m}{R}\dot{U} + \frac{1}{k_m}M_{ext}, \end{cases} \quad (6)$$

therefore its modeling scheme is equal to one which is demonstrated by figure 8.

Modeling scheme of DC motor

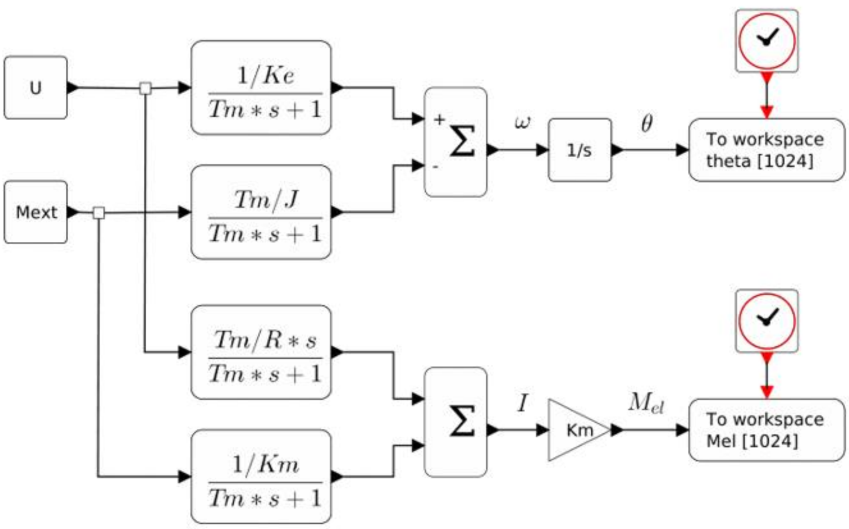


Figure 8. Modeling scheme of DC motor.

Control of DC motor using PID controller

Control and modeling of mobile robots
Control of DC motor using PID regulator

Alexander A. Kapitonov

Some basics of control theory

In a control theory all systems are considered as a single object or a "box" which has some number of input and output signals.

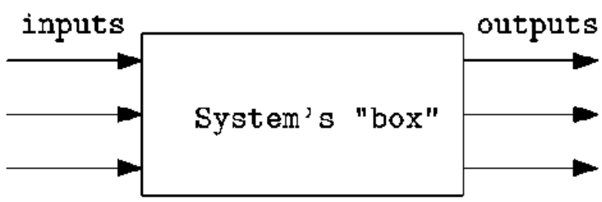


Figure 1. One of a possible representation of every system in a control theory.

input signals — some impacts which change system state

output signals — some physical quantities which describe system state

Some basics of control theory

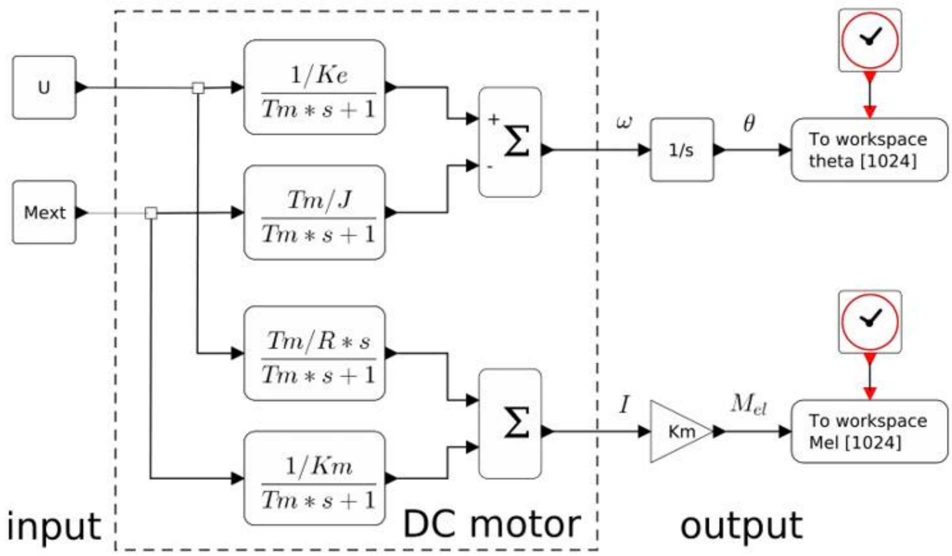


Figure 2. Structure of the model of a DC motor.

Some basics of control theory

Some important definitions:

Control is a process of changing in a desired way values of some output signals using some input signals.

Controller is a special device and/or algorithm which creates required input signals.

Methods of control:

- forward
- using feedback

Some basics of control theory

Forward control — a method of control when a controller doesn't use information about values of system's output signals.

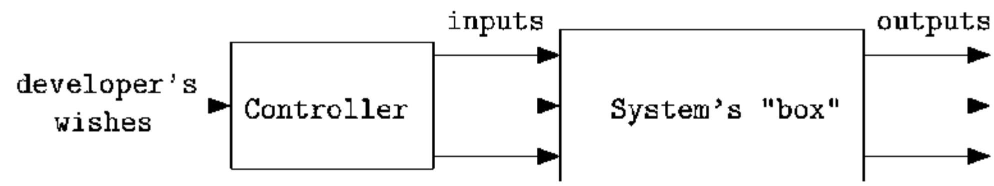


Figure 3. Scheme of forward control.

Some basics of control theory

Control with feedback — a method of control when a controller use information about values of system's output signals.

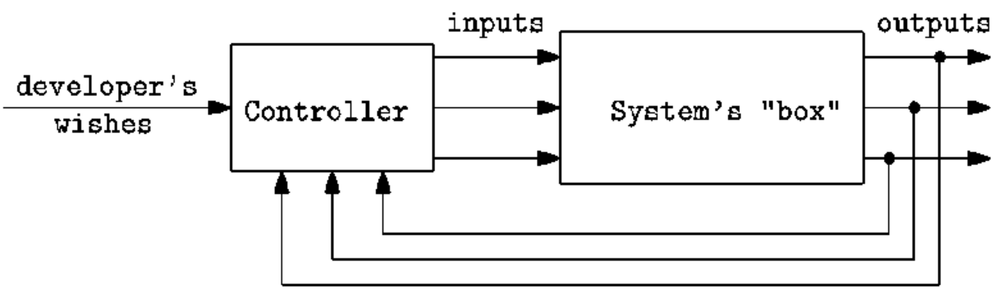


Figure 4. Scheme of control with feedback.

PID controller

PID controller is an algorithm of feedback control which calculate value for input signal in accordance to the formulas:

$$e(t) = x_d(t) - x(t), \quad (1)$$

$$u = K_p \cdot e + K_i \cdot \int e\,dt + K_d \dot{e}, \quad (2)$$

where x — controllable output signal; x_d — desired value of signal; e — error of control; u — used system's input signal; K_p, K_i, K_d — constant coefficients of PID controller.

PID controller

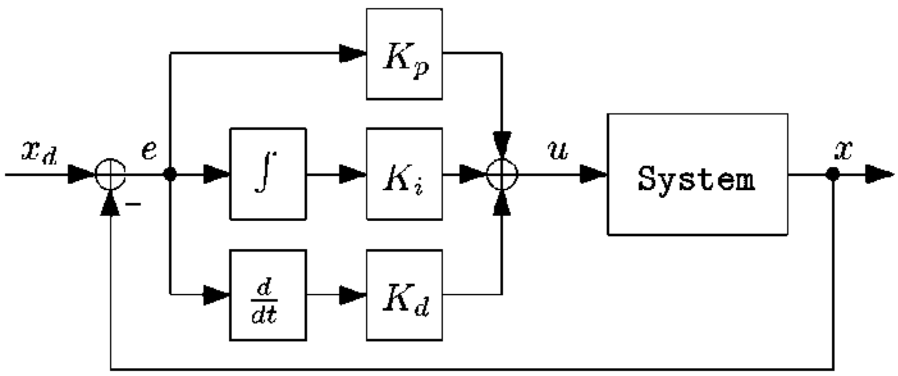


Figure 5. Scheme of PID controller sctructure.

PID controller

P controlller or a proportional piece of PID which is calculated as

$$u = K_p \cdot e, \tag{3}$$

does the main part of a controller's job;

I controlller or a piece of PID with integral which is calculated as

$$u = K_i \cdot \int e\, dt, \tag{4}$$

prevents errors (makes e is being equal to 0);

D controlller or a piece of PID with derivative which is calculated as

$$u = K_d \cdot \dot{e}, \tag{5}$$

dampens oscillations.

PID controller

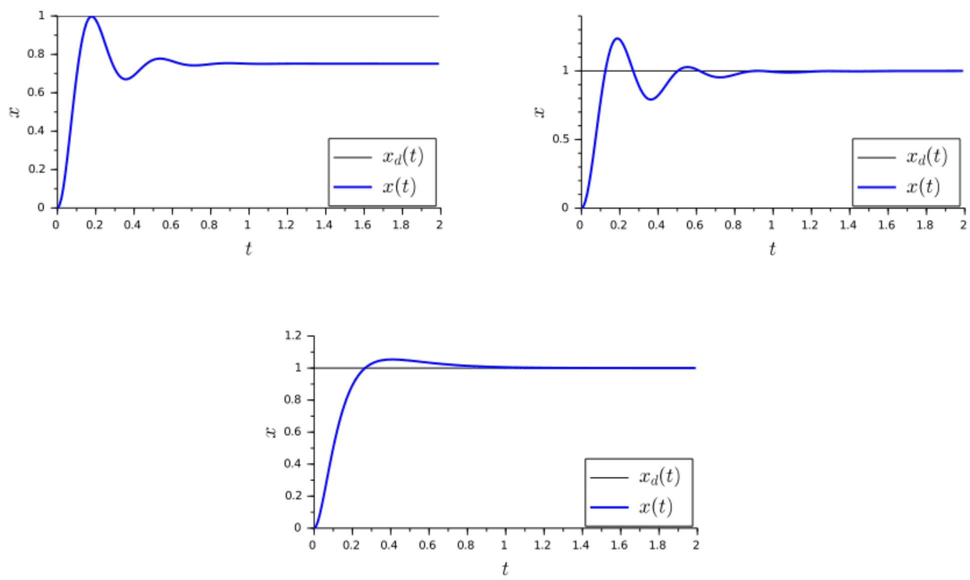


Figure 6. System with P, PI and PID controller respectively.

PID controller

Methods of tuning controller's coefficients:

- calculations using mathematical model of a controllable object;
- setting with according to one of a special algorithm;
- fully manual setting.

Ziegler–Nichols method

Algorithm of tuning values for coefficients of PID controller:

1. make K_i and K_d is being equal to 0;
2. increase value of K_p until $x(t)$ starts making undamped oscillations; remember this value of K_p as K_u and a period of the oscillations as T_u;
3. calculate coefficients of PID controller using these formulas:

$$K_p = 0.6K_u, \qquad K_i = \frac{2K_p}{T_u}, \qquad K_d = \frac{K_pT_u}{8}. \tag{6}$$

Ziegler–Nichols method

This method's strengths:

- it is quite simple.

This method's weaknesses:

- it doesn't work for all systems;
- it doesn't give the best value of coefficients.

Numerical methods

$$\int_{t_1}^{t_2} f\,dt = S_{1,2} \tag{7}$$

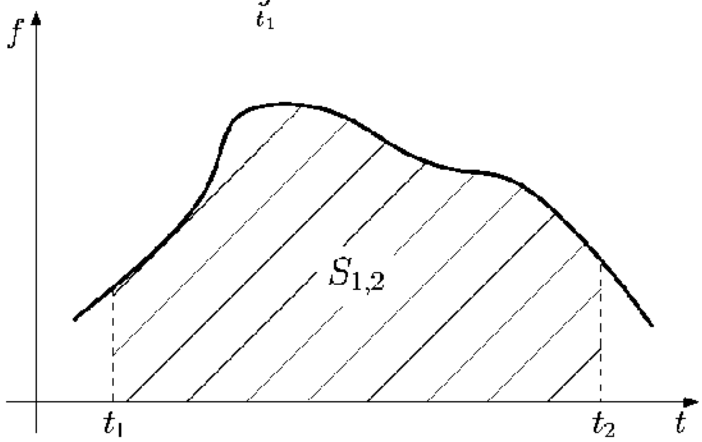


Figure 7. Geometry meaning of integrals.

Numerical methods

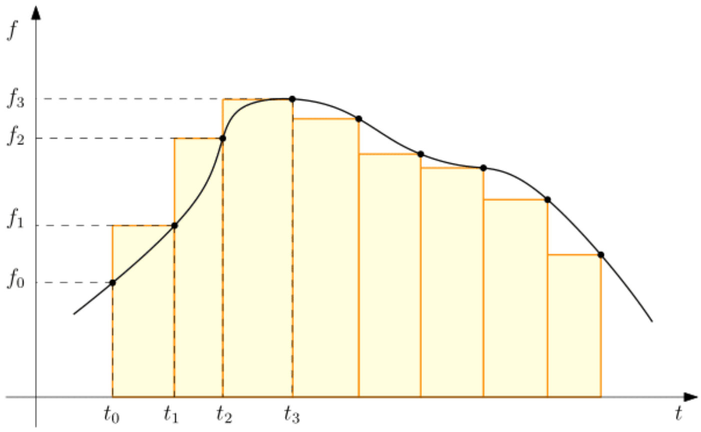


Figure 8. One of numerical methods for calculating value of integral.

$$\int_{t_m}^{t_n} f\,dt \approx \sum_{i=m+1}^{n} f_i(t_i - t_{i-1}), \quad m < n, \ \ m, n \in \mathbb{Z} \tag{8}$$

Numerical methods

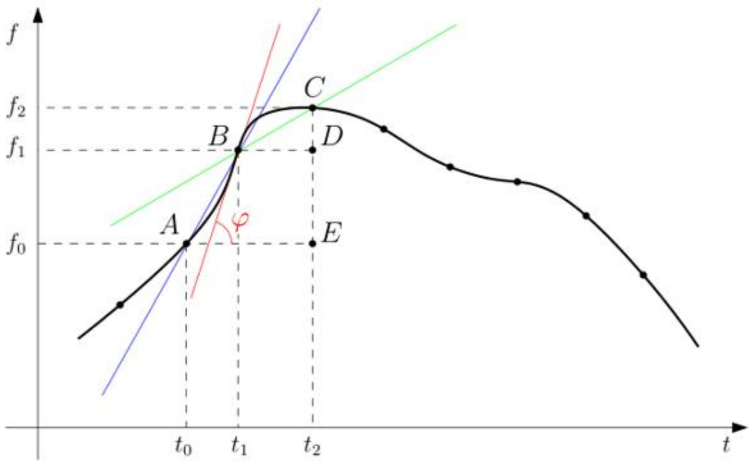


Figure 9. Numerical methods for derivative calculating.

$$f'(t_1) = \lim_{\Delta t \to 0} \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t} = \operatorname{tg}\varphi, \quad f'(t_1) \approx \frac{f_1 - f_0}{t_1 - t_0} = \operatorname{tg}\angle BAE$$

A controller for to-point motion for a mobile robot with differential drive type

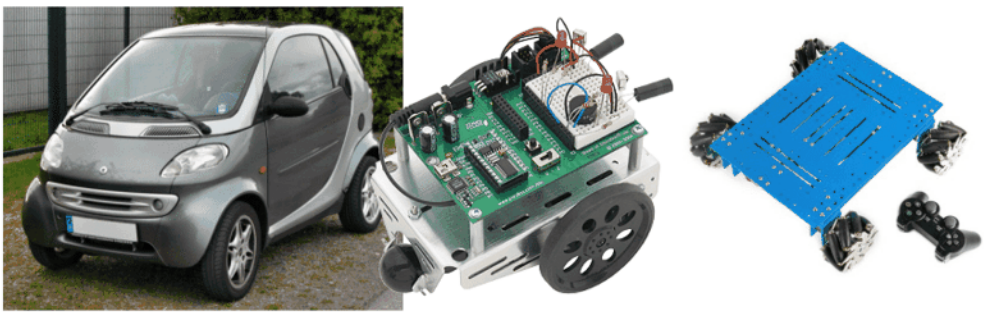
Control and modeling of mobile robots

A controller for to-point motion for a mobile robot with differential drive type

Alexander A. Kapitonov

Robots' drive types

Drive type	Controllable velocities
Car-like type	v_x, $\omega(v_x)$
Differential	v_x, ω
Omnidirectional	v_x, v_y, ω



a) b) c)

Figure 1. Examples of "robots" with different drive types: a—car-like, b—differential, c—omnidirectional

General view of the robot

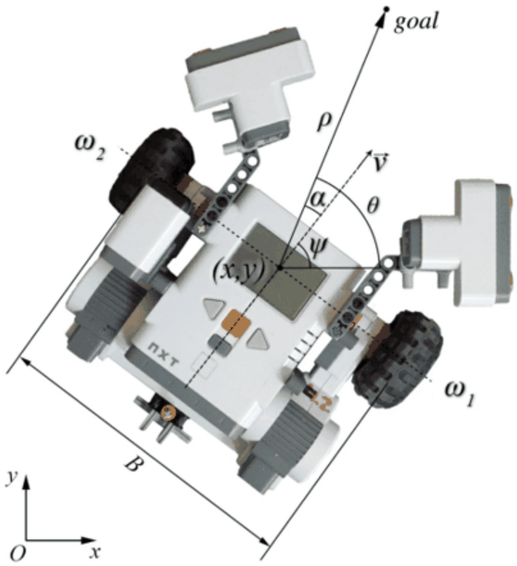


Figure 2. General view of a considered robot.

Structure of the control system

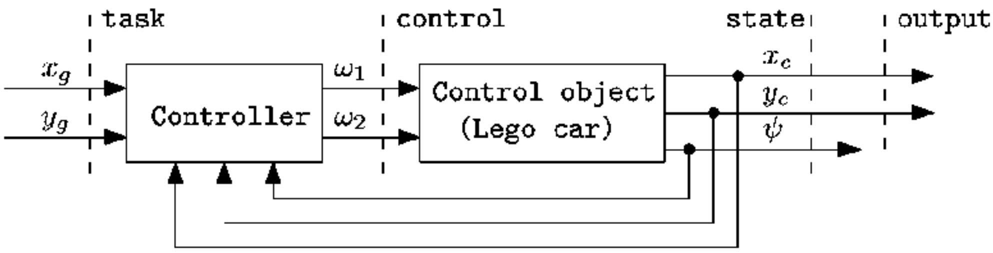


Figure 3. Structure of the control system.

x_g, y_g — coordinates of goal point;

ω_1, ω_2 — angular velocities of robot's wheels;

x_c, y_c, ψ coordinates and rotation angle of the robot.

Mathematical model of the robot

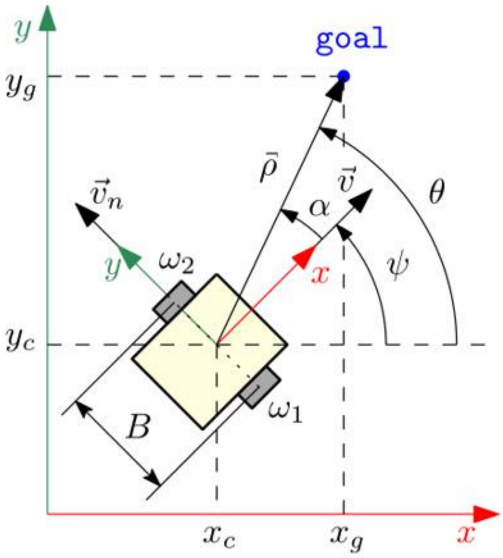


Figure 4. Useful drawing.

Kinematic model:

$$\begin{cases} \dot{x}_c = |\vec{v}| \cos\psi \\ \dot{y}_c = |\vec{v}| \sin\psi \\ \dot{\psi} = \omega \end{cases} \quad (1)$$

where

$$|\vec{v}| = R \cdot \frac{\omega_1 + \omega_2}{2}, \quad (2)$$

$$\omega = \frac{R}{B} \cdot (\omega_1 - \omega_2), \quad (3)$$

where R — wheel radius.

Mathematical model of the robot

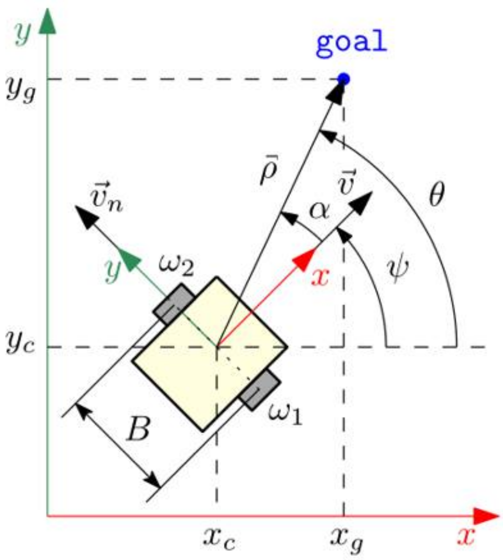


Figure 5. Useful drawing.

Some important variables:

$$\vec{\rho} = \left\{ x_g - x_c \quad y_g - y_c \right\}, \quad (4)$$

$$\theta = \arctan \frac{y_g - y_c}{x_g - x_c}, \quad (5)$$

$$\alpha = \theta - \psi, \quad (6)$$

$$|\vec{v}_n| = |\vec{v}|. \quad (7)$$

Mathematical model of the robot

$$|\vec{\rho}| = \sqrt{(x_g - x_c)^2 + (y_g - y_c)^2} \tag{8}$$

$$\begin{aligned}
\frac{d|\vec{\rho}|}{dt} &= \frac{1}{2\sqrt{(x_g - x_c)^2 + (y_g - y_c)^2}} \cdot \left((x_g - x_c)^2 + (y_g - y_c)^2\right)' = \\
&= \frac{1}{2|\vec{\rho}|}(-2\dot{x}_c(x_g - x_c) - 2\dot{y}_c(y_g - y_c)) = \\
&= -\frac{1}{|\vec{\rho}|} \cdot \{\dot{x}_c \;\; \dot{y}_c\} \cdot \{x_g - x_c \;\; y_g - y_c\} = -\frac{1}{|\vec{\rho}|} \cdot \vec{v} \cdot \vec{\rho} = -|\vec{v}|\cos\alpha
\end{aligned} \tag{9}$$

$$\dot{\alpha} = \dot{\theta} - \dot{\psi} \tag{10}$$

Mathematical model of the robot

$$\begin{aligned}
\dot{\theta} &= \left(\arctan\frac{y_g - y_c}{x_g - x_c}\right)' = \frac{1}{1 + \left(\frac{y_g - y_c}{x_g - x_c}\right)^2} \cdot \left(\frac{y_g - y_c}{x_g - x_c}\right)' = \\
&= \frac{(x_g - x_c)^2}{(x_g - x_c)^2 + (y_g - y_c)^2} \cdot \frac{-\dot{y}_c(x_g - x_c) + \dot{x}_c(y_g - y_c)}{(x_g - x_c)^2} = \\
&= \frac{\{-\dot{y}_c \;\; \dot{x}_c\} \cdot \{x_g - x_c \;\; y_g - y_c\}}{|\vec{\rho}|^2} = \frac{\vec{v}_n \cdot \vec{\rho}}{|\vec{\rho}|^2} = \\
&= \frac{|\vec{v}_n|\cos(90^\circ - \alpha)}{|\vec{\rho}|} = \frac{|\vec{v}|\sin\alpha}{|\vec{\rho}|}
\end{aligned} \tag{11}$$

$$\dot{\alpha} = \frac{|\vec{v}|\sin\alpha}{|\vec{\rho}|} - \omega \tag{12}$$

Mathematical model of the robot

Robot's mathematical model:

$$\begin{cases} \dfrac{d|\vec{\rho}|}{dt} = -|\vec{v}|\cos\alpha \\ \dfrac{d\alpha}{dt} = \dfrac{|\vec{v}|\sin\alpha}{|\vec{\rho}|} - \omega \end{cases} \quad \text{or} \quad \dot{x} = f(x), \text{ where } x = \begin{bmatrix} |\vec{\rho}| \\ \alpha \end{bmatrix} \tag{13}$$

Let's use for it this control law:

$$\begin{cases} |\vec{v}| = v_{max} \cdot \tanh|\vec{\rho}| \cdot \cos\alpha \\ \omega = K_\omega \alpha + v_{max} \cdot \dfrac{\tanh|\vec{\rho}|}{|\vec{\rho}|} \cdot \sin\alpha \cdot \cos\alpha \end{cases} \tag{14}$$

where v_{max} and K_ω are constant positive coefficients.

Mathematical model of the robot

Some theoretical information:

- Stability is an ability of a controlled system to run to particular state and stay in it.
- For checking system for stability Lyapunov functions are used.
- If time derivative of Lyapunov functions for considered system is always negative, the system is stable.

Mathematical model of the robot

Possible Lyapunov function for our system:

$$V(x) = \frac{1}{2}|\vec{\rho}|^2 + \frac{1}{2}\alpha^2 \tag{15}$$

Its derivative:

$$\frac{dV}{dt} = \frac{d|\vec{\rho}|}{dt} \cdot |\vec{\rho}| + \frac{d\alpha}{dt} \cdot \alpha = -|\vec{v}||\vec{\rho}| \cos\alpha + \alpha \left(\frac{|\vec{v}| \sin\alpha}{|\vec{\rho}|} - \omega \right) \tag{16}$$

or after using equations (14) for control law:

$$\frac{dV}{dt} = -v_{max} \cdot |\vec{\rho}| \cdot \tanh\ \vec{\rho}\ \cdot \cos^2\alpha - K_\omega \alpha^2 < 0. \tag{17}$$

Due to $\dot{V}$ is always negative the system is stable.

Mathematical model of the robot

Note that:

- angular speeds of robot's wheels can be found using these formulas:

$$\omega_1 = \frac{1}{R} \cdot (2|\vec{v}| + B\omega), \qquad \omega_2 = \frac{1}{R} \cdot (2|\vec{v}| - B\omega). \tag{18}$$

- in the steady state angular speeds of robot's motors are proportional to voltages which are applied to them; so we will make the latters are being proportional to values obtained from equations (18).

Sources for pictures

- slide 2:
  - https://en.wikipedia.org/wiki/Car
  - https://www.parallax.com/product/boe-bot-robot
  - http://www.makeblock.com/mecanum-wheel-robot-kit

ITMO UNIVERSITY
Faculty of Control systems and robotics

- per. Grivtsova, 14,
  Saint Petersburg, Russia, 190000

  +7 (812) 595-41-28
  csi.ifmo.ru/en/